

# On the Coupling Constants, Geometric Probability and Shilov Boundaries

Carlos Castro\*

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**Abstract.** By recurring to Geometric Probability methods it is shown that the coupling constants,  $\alpha_{EM}, \alpha_W, \alpha_C$ , associated with the Electromagnetic, Weak and Strong (color) force are given by the *ratios* of measures of the sphere  $S^2$  and the Shilov boundaries  $Q_3 = S^2 \times RP^1$ , *squashed*  $S^5$ , respectively, with respect to the Wyler measure  $\Omega_{Wyler}[Q_4]$  of the Shilov boundary  $Q_4 = S^3 \times RP^1$  of the poly-disc  $D_4$  ( 8 real dimensions). The latter measure  $\Omega_{Wyler}[Q_4]$  is linked to the geometric coupling strength  $\alpha_G$  associated to the gravitational force. In the conclusion we discuss briefly other approaches to the determination of the physical constants, in particular, a program based on the Mersenne primes  $p$ -adic hierarchy. The most important conclusion of this work is the role played by higher dimensions in the determination of the coupling constants from pure geometry and topology alone and which does *not* require to invoke the anthropic principle .

## 1 GEOMETRIC PROBABILITY

Geometric Probability [1] is the study of the probabilities involved in geometric problems, e.g., the distributions of length, area, volume, etc. for geometric objects under stated conditions. One of the most famous problem is the Buffon's Needle Problem of finding the probability that a needle of length  $l$  will land on a line, given a floor with equally spaced parallel lines a distance  $d$  apart. The problem was first posed by the French naturalist Buffon in 1733. For  $l < d$  the probability is

$$P = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{l|\cos(\theta)|}{d} = \frac{4l}{2\pi d} \int_0^{\pi/2} \cos(\theta) = \frac{2l}{\pi d} = \frac{2ld}{\pi d^2}. \quad (1.1)$$

Hence, the Geometric Probability is essentially the *ratio* of the areas of a rectangle of length  $2d$ , and width  $l$  and the area of a circle of radius  $d$  . For  $l > d$ , the solution is slightly more complicated [1]. The Buffon needle problem provides with a numerical experiment that determines the value of  $\pi$  empirically. Geometric Probability is a vast field with profound connections to Stochastic Geometry.

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\*Center for Theoretical Studies of Physical Systems, Clark Atlanta University, Atlanta GA. 30314; e-mail : castro@ctsps.cau.edu

Feynman long ago speculated that the fine structure constant may be related to  $\pi$ . This is the case as Wyler found long ago [2]. We will base our derivation of the fine structure constant based on Feynman's physical interpretation of the electron's charge as the probability amplitude that an electron emits (or absorbs) a photon. The clue to evaluate this probability within the context of Geometric Probability theory is provided by the electron self-energy diagram. Using Feynman's rules, the self-energy  $\Sigma(p)$  as a function of the electron's incoming (outgoing) energy-momentum  $p_\mu$  is given by the integral involving the photon and electron propagator along the internal lines

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i}{\gamma^\rho(p_\rho - k_\rho) - m} \frac{-ig_{\mu\nu}}{k^2} \gamma^\nu. \quad (1.2)$$

The integral is taken with respect to the values of the photon's energy-momentum  $k^\mu$ . By inspection one can see that the electron self-energy is proportional to the fine structure constant  $\alpha_{EM} \sim e^2$ , the square of the probability amplitude (in natural units of  $\hbar = c = 1$ ) and physically represents the electron's emission of a virtual photon (off-shell,  $k^2 \neq 0$ ) of energy-momentum  $k_\rho$  at a given moment, followed by an absorption of this virtual photon at a later moment.

Based on this physical picture of the electron self-energy graph, we will evaluate the Geometric Probability that an electron emits a photon at  $t = -\infty$  (infinite past) and re-absorbs it at a much later time  $t = +\infty$  (infinite future). The off-shell (virtual) photon associated with the electron self-energy diagram *asymptotically* behaves on-shell at the very moment of emission ( $t = -\infty$ ) and absorption ( $t = +\infty$ ). However, the photon can remain off-shell in the intermediate region between the moments of emission and absorption by the electron.

The topology of the boundaries (at conformal infinity) of the past and future light-cones are spheres  $S^2$  (the celestial sphere). This explains why the (Shilov) boundaries are essential mathematical features to understand the geometric derivation of all the coupling constants. In order to describe the physics at infinity we will recur to Penrose's ideas [11] of conformal compactifications of Minkowski spacetime by attaching the light-cones at conformal infinity. Not unlike the one-point compactification of the complex plane by adding the points at infinity leading to the Gauss-Riemann sphere. The conformal group leaves the light-cone fixed and it does not alter the causal properties of spacetime despite the rescalings of the metric. The topology of the conformal compactification of real Minkowski spacetime  $\bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$  is precisely the same as the topology of the Shilov boundary  $Q_4$  of the 4 complex-dimensional poly-disc  $D_4$ . The action of the discrete group  $Z_2$  amounts to an antipodal identification of the future null infinity  $\mathcal{I}^+$  with the past null infinity  $\mathcal{I}^-$ ; and the antipodal identification of the past timelike infinity  $i^-$  with the future timelike infinity  $i^+$ , where the electron emits, and absorbs the photon, respectively.

Shilov boundaries of homogeneous (symmetric spaces) complex domains,  $G/K$  [8], [9], [10] are not the same as the ordinary topological boundaries (except in some special cases). The reason being that the action of the isotropy group  $K$  of the origin is not necessarily *transitive* on the ordinary topological boundary. Shilov boundaries are the minimal subspaces of the ordinary topological boundaries which implement the Maldacena-'t Hooft-Susskind *Holographic* principle [14] in the sense that the holomorphic data in the interior (bulk) of the domain is fully determined by the holomorphic data on the Shilov

boundary. The latter has the property that the maximum modulus of any holomorphic function defined on a domain is attained at the Shilov boundary.

For example, the poly-disc  $D_4$  of 4 complex dimensions is an 8 real-dim Hyperboloid of constant negative scalar curvature that can be identified with the conformal relativistic *curved* phase space associated with the electron (a particle) moving in a 4D Anti de Sitter space  $AdS_4$ . The poly-disc is a Hermitian symmetric homogeneous coset space associated with the 4D conformal group  $SO(4, 2)$  since  $D_4 = SO(4, 2)/SO(4) \times SO(2)$ . Its Shilov boundary  $Shilov(D_4) = Q_4$  has precisely the *same* topology as the 4D conformally compactified real Minkowski spacetime  $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$ . For more details about Shilov boundaries, the conformal group, future tubes and holography we refer to the article by Gibbons [13] and [8], [17].

The role of the conformal group in Gravity in these expressions (besides the holographic bulk/boundary  $AdS/CFT$  duality correspondence [14]) stems from the MacDowell Mansouri-Chamseddine-West formulation of Gravity based on the conformal group  $SO(3, 2)$  which has the same number of 10 generators as the 4D Poincare group. The 4D vielbein  $e_\mu^a$  which gauges the spacetime translations is identified with the  $SO(3, 2)$  generator  $A_\mu^{[a5]}$ , up to a crucial scale factor  $R$ , given by the size of the Anti de Sitter space (de Sitter space) throat. It is known that the Poincare group is the Wigner-Inonu group contraction of the de Sitter Group  $SO(4, 1)$  after taking the throat size  $R = \infty$ . The spin-connection  $\omega_\mu^{ab}$  that gauges the Lorentz transformations is identified with the  $SO(3, 2)$  generator  $A_\mu^{[ab]}$ . In this fashion, the  $e_\mu^a, \omega_\mu^{ab}$  are encoded into the  $A_\mu^{[mn]}$   $SO(3, 2)$  gauge fields, where  $m, n$  run over the group indices 1, 2, 3, 4, 5. A word of caution, Gravity is a gauge theory of the full diffeomorphisms group which is infinite-dimensional and which includes the translations. Therefore, strictly speaking gravity is not a gauge theory of the Poincare group. The Ogirovetsky theorem shows that the diffeomorphisms algebra in 4D can be generated by an infinity of *nested* commutators involving the  $GL(4, R)$  and the 4D Conformal Group  $SO(4, 2)$  generators.

In [18] we have shown why the MacDowell-Mansouri-Chamseddine-West formulation of Gravity, with a cosmological constant and a topological Gauss-Bonnet invariant term, can be obtained from an action inspired from a BF-Chern-Simons-Higgs theory based on the conformal  $SO(3, 2)$  group. The  $AdS_4$  space is a natural vacuum of the theory. The vacuum energy density was *derived* to be precisely the geometric-mean between the UV Planck scale and the IR throat size of de Sitter (Anti de Sitter) space. Setting the throat size to coincide with the future horizon scale (of an accelerated de Sitter Universe) given by the Hubble scale (today)  $R_H$ , the geometric mean relationship yields the observed value of the vacuum energy density  $\rho \sim (L_P)^{-2}(R_H)^{-2} = (L_P)^{-4}(L_P^2/R_H^2) \sim 10^{-120} M_{Planck}^4$ . Nottale [23] gave a different argument to explain the small value of  $\rho$  based on Scale Relativistic arguments. It was also shown in [18] why the Euclideanized  $AdS_{2n}$  spaces are  $SO(2n-1, 2)$  instantons solutions of a non-linear sigma model obeying a double self duality condition.

A typical objection to the possibility of being able to derive the values of the coupling constants, from pure thought alone, is that there are an infinite number of possible analytical expressions that accurately reproduce the values of the couplings within the experimental error bounds. However, this is not our case because once the gauge groups  $U(1), SU(2), SU(3)$  are known there are *unique* expressions stemming from Geometric Probability which furnish the values of the couplings. Another objection is that it is a mean-

ingless task to try to derive these couplings because these are not constants per se but vary with respect to the energy scale. The running of the coupling constants is an *artifact* of the perturbative Renormalization Group program. We will see that the values of the couplings derived from Geometric Probability are precisely those values that correspond to the natural physical scales associated with the EM, Weak and Strong forces.

Another objection is that physical measurements of irrational numbers are impossible because there are always experimental limitations which rule out the possibility of actually measuring the *infinite* number of digits of an irrational number. This experimental constraint does not exclude the possibility of deriving exact expressions based on  $\pi$  as we shall see. We should not worry also about obtaining numerical values within the error bars in the table of the coupling constants since these numbers are based on the values of *other* physical constants; i.e. they are based on the particular *consensus* chosen for all of the other physical constants.

In our conventions,  $\alpha_{EM} = e^2/4\pi = 1/137.036\dots$  in the natural units of  $\hbar = c = 1$ , and the quantities  $\alpha_{weak}, \alpha_{color}$  are the Geometric Probabilities  $\tilde{g}_w^2, \tilde{g}_c^2$ , after *absorbing* the factors of  $4\pi$  of the conventional  $\alpha_w = (g_w^2/4\pi), \alpha_c = (g_c^2/4\pi)$  definitions used in the Renormalization Group (RG) program.

## 2 THE FINE STRUCTURE CONSTANT

In order to define the Geometric Probability associated with this process of the electron's emission of a photon at  $i^-$  ( $t = -\infty$ ), followed by an absorption at  $i^+$  ( $t = +\infty$ ), we must take into account the important fact that the photon is on-shell  $k^2 = 0$  *asymptotically* (at  $t = \pm\infty$ ), but it can move off-shell  $k^2 \neq 0$  in the intermediate region which is represented by the *interior* of the 4D conformally compactified real Minkowski spacetime which agrees with the Shilov boundary of  $D_4$  (the four-complex-dimensional poly-disc )  $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$ .  $Q_4$  has four-real-dimensions which is half the real-dimensions of  $D_4$  ( $2 \times 4 = 8$ ).

The measure associated with the celestial spheres  $S^2$  (associated with the future/past light-cones) at timelike infinity  $i^+, i^-$ , respectively, is  $V(S^2) = 4\pi r^2 = 4\pi$  ( $r = 1$ ). Thus, the *net* measure corresponding to the two celestial spheres  $S^2$  at timelike infinity  $i^\pm$  requires an overall factor of 2 giving  $2V(S^2) = 8\pi$  ( $r = 1$ ). The Geometric Probability is defined by the ratio of the (dimensionless volumes) measures associated with the celestial spheres  $S^2$  at  $i^+, i^-$  timelike infinity, where the photon moves on-shell, relative to the Wyler measure  $\Omega_{Wyler}[Q_4]$  associated with the full *interior* region of the conformally compactified 4D Minkowski space  $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$ , where the massive electron is confined to move, as it propagates from  $i^-$  to  $i^+$ , (and *off-shell* photons can also live in ) :

$$\alpha_{EM} = \frac{2V(S^2)}{\Omega_{Wyler}[Q_4]} = \frac{8\pi}{\Omega_{Wyler}[Q_4]} = \frac{1}{137.03608\dots} \quad (2.1a)$$

after inserting the Wyler measure

$$\Omega_{Wyler}[Q_4] = \frac{V(S^4) V(Q_5)}{[V(D_5)]^{1/4}} = \left(\frac{8\pi^2}{3}\right)\left(\frac{8\pi^3}{3}\right)\left(\frac{\pi^5}{2^4 \times 5!}\right)^{-1/4}. \quad (2.1b)$$

The Wyler measure  $\Omega_{Wyler}[Q_4]$  [2] is *not* the standard measure (dimensionless volume)  $V(Q_4) = 2\pi^3$  calculated by Hua [9] but requires some elaborate procedure.

It was realized by Smith [4] that the presence of the Wyler measure in the expression for  $\alpha_{EM}$  given by eq-(2-1) was consistent with Wheeler ideas that the observed values of the coupling constants of the Electromagnetic, Weak and Strong Force can be obtained if the geometric force strengths (measures related to volumes of complex homogenous domains associated with the  $U(1)$ ,  $SU(2)$ ,  $SU(3)$  groups, respectively ) are divided by the geometric force strength of Gravity  $\alpha_G$  (related to the  $SO(3, 2)$  MMCW Gauge Theory of Gravity ) and which is not the same as the 4D Newton's gravitational constant  $G_N \sim m_{Planck}^{-2}$ .

Furthermore, the expression for  $\Omega_{Wyler}[Q_4]$  is also consistent with the Kaluza-Klein compactification procedure of obtaining Maxwell's EM in 4D from pure Gravity in 5D. In particular, in order to evaluate the Wyler measure  $\Omega_{Wyler}[Q_4]$  one requires to embed  $D_4$  into  $D_5$  because the Shilov boundary space  $Q_4 = S^3 \times RP^1$  is *not* adequate enough to implement the action of the  $SO(5)$  group, the compact version of the Anti de Sitter Group  $SO(3, 2)$  that is required in the MacDowell-Mansouri-Chamseddine-West (MMCW)  $SO(3, 2)$  Gauge formulation of Gravity. However, the Shilov boundary of  $D_5$  given by  $Q_5 = S^4 \times RP^1$  is adequate enough to implement the action of  $SO(5)$  via isometries (rotations) on the internal symmetry space  $S^4 = SO(5)/SO(4)$ . This justifies the embedding procedure of  $D_4 \rightarrow D_5$ .

The 5 complex-dimensional poly-disc  $D_5 = SO(5, 2)/SO(5) \times SO(2)$  is the 10 real-dim Hyperboloid  $\mathcal{H}^{10}$  corresponding to the relativistic curved phase space of a particle moving in 5D Anti de Sitter Space  $AdS_5$  . The Shilov boundary  $Q_5$  of  $D_5$  has 5 real dimensions (half of the 10-real-dim of  $D_5$ ). One cannot fail to notice that the hyperboloid  $\mathcal{H}^{10}$  can be embedded in the 11-dim pseudo-Euclidean  $R^{9,2}$  space, with two-time like directions. This is where 11-dim lurks into our construction.

Having displayed Wyler's expression of the fine structure constant  $\alpha_{EM}$  in terms of the ratio of dimensionless measures , we shall present a Fiber Bundle (a sphere bundle fibration over a complex homogeneous domain) derivation of the Wyler expression based on the bundle  $S^4 \rightarrow E \rightarrow D_5$ . The physical reasoning behind the role of the Shilov boundaries is the following. Wyler has shown that it is possible to map an unbounded physical domain (the interior of the future light cone) onto the interior of a homogenous bounded domain without losing the causal structure and on which there exists also a complex structure. Massless gauge bosons live on the lightcone, a null boundary in Minkowski spacetime. Thus, upon performing the Wyler map, the gauge bosons (photon in this case) are confined to live on the Shilov boundary while the electron is confined to the *interior* of the homogenous bounded domain. We will show below why the propagation (via the determinant of the Feynman propagator) of the electron through the *interior* of the domain  $D_5$  is what accounts for the "obscure" factor  $V(D_5)^{1/4}$  in Wyler's formula for  $\alpha_{EM}$ .

We begin by explaining why Wyler's measure  $\Omega_{Wyler}[Q_4]$  in eq-(2-1) corresponds to the measure of a  $S^4$  bundle fibered over the base curved-space  $D_5 = SO(5, 2)/SO(5) \times SO(2)$  and *weighted* by a factor of  $V(D_5)^{-1/4}$ . This  $S^4 \rightarrow E \rightarrow D_5$  bundle is linked to the MMCW  $SO(3, 2)$  Gauge theory formulation of gravity and explains the essential role of gravity in Wyler's formula [4] corroborating Wheeler's ideas that one must normalize the geometric force strengths with respect to gravity in order to obtain the coupling constants. The subgroup  $H = SO(5)$  of the isotropy group (at the origin)  $K = SO(5) \times SO(2)$  acts naturally on the Fibers  $F = S^4 = SO(5)/SO(4)$ , the internal symmetric space, via isometries (rotations). Locally, and only locally, the Fiber bundle  $E$  is the product

$D_5 \times S^4$ . The restriction of the Fiber bundle  $E$  to the Shilov boundary  $Q_5$  is written as  $E|_{Q_5}$  and *locally* is the product of  $Q_5 \times S^4$ , but this is *not* true globally unless the fiber bundle admits a global section (the bundle is trivial). For this reason the volume  $V(E|_{Q_5})$  does not necessary always factorize as  $V(Q_5) \times V(S^4)$ . Setting aside this subtlety, we shall pursue a more physical route, already suggested by Wyler <sup>1</sup>, to explain the origin of the "obscure normalization" factor  $V(D_5)^{1/4}$  in Wyler's measure  $\Omega_{Wyler}[Q_4] = (V(S^4) \times V(Q_5)/V(D_5)^{1/4})$ , which suggests that the volumes may not factorize.

The Feynman propagator of a massive scalar particle (inverse of the Klein-Gordon operator)  $(D_\mu D^\mu - m^2)^{-1}$  corresponds to the *kernel* in the Feynman path integral that in turn is associated with the Bergman kernel  $K_n(z, z')$  of the complex homogenous domain  $D_n$  which is proportional to the Bergman constant  $k_n \equiv 1/V(D_n)$ .

$$(D_\mu D^\mu - m^2)^{-1}(x^\mu) = \frac{1}{(2\pi\mu)^D} \int d^D p \frac{e^{-ip_\mu x^\mu}}{p^2 - m^2 + i\epsilon} \leftrightarrow K_n(\mathbf{z}, \bar{\mathbf{z}}') = \frac{1}{V(D_n)} (1 - \mathbf{z}\bar{\mathbf{z}}')^{-2n}. \quad (2.2)$$

where we have introduced a momentum scale  $\mu$  to match units in the Feynman propagator expression, and the Bergman Kernel  $K_n(\mathbf{z}, \bar{\mathbf{z}}')$  of  $D_n$  whose dimensionless entries are  $\mathbf{z} = (z_1, z_2, \dots, z_n)$ ,  $\mathbf{z}' = (z'_1, z'_2, \dots, z'_n)$  is given as

$$K_n(\mathbf{z}, \bar{\mathbf{z}}') = \frac{1}{V(D_n)} (1 - \mathbf{z}\bar{\mathbf{z}}')^{-2n} \quad (2.3a)$$

$V(D_n)$  is the dimensionless Euclidean volume found by Hua  $V(D_n) = (\pi^n/2^{n-1}n!)$  and satisfies the reproducing and normalization properties

$$f(z) = \int_{D_n} f(\xi) K_n(z, \xi) d^n \xi d^n \bar{\xi}. \quad \int_{D_n} K_n(z, \bar{z}) d^n z d^n \bar{z} = 1. \quad (2.3b)$$

The *key* result that can be inferred from the Feynman propagator (kernel)  $\leftrightarrow$  Bergman kernel  $K_n$  correspondence, when  $\mu = 1$ , is the  $(2\pi)^{-D} \leftrightarrow (V(D_n))^{-1}$  correspondence; i.e. the fundamental hyper-cell in momentum space  $(2\pi)^D$  (when  $\mu = 1$ ) corresponds to the dimensionless volume  $V(D_n)$  of the domain, where  $D = 2n$  real dimensions. The regularized vacuum-to-vacuum amplitude of a free *real* scalar field is given in terms of the zeta function  $\zeta(s) = \sum_i \lambda_i^{-s}$  associated with the eigenvalues of the Klein-Gordon operator by

$$Z = \langle 0 | 0 \rangle = \sqrt{\det (D_\mu D^\mu - m^2)^{-1}} \sim \exp \left[ \frac{1}{2} \frac{d\zeta}{ds}(s=0) \right]. \quad (2.4)$$

In case of a *complex* scalar field we have to *double* the number of degrees of freedom, the amplitude then factorizes into a product and becomes  $Z = \det (D_\mu D^\mu - m^2)^{-1}$ .

Since the Dirac operator  $\mathcal{D} = \gamma^\mu D_\mu + m$  is the "square-root" of the Klein-Gordon operator  $\mathcal{D}^\dagger \mathcal{D} = D_\mu D^\mu - m^2 + \mathcal{R}$  ( $\mathcal{R}$  is the scalar curvature of spacetime that is zero in Minkowski space) we have the numerical correspondence

$$\sqrt{\det (\mathcal{D})^{-1}} = \sqrt{\det (D_\mu D^\mu - m^2)^{-1/2}} = \sqrt{\sqrt{\det (D_\mu D^\mu - m^2)^{-1}}} \leftrightarrow k_n^{1/4} = \left( \frac{1}{V(D_n)} \right)^{1/4}. \quad (2.5)$$

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<sup>1</sup>We thank Frank (Tony) Smith for this information

because  $\det \mathcal{D}^\dagger = \det \mathcal{D}$ , and

$$\det \mathcal{D} = e^{\text{trace } \ln \mathcal{D}} = e^{\text{trace } \ln (D_\mu D^\mu - m^2)^{1/2}} = e^{\frac{1}{2} \text{tr } \ln (D_\mu D^\mu - m^2)} = \sqrt{\det (D_\mu D^\mu - m^2)}. \quad (2.6)$$

The vacuum-to-vacuum amplitude of a *complex* Dirac field  $\Psi$  (a fermion, the electron) is  $Z = \det (\gamma^\mu D_\mu + m) = \det \mathcal{D} \sim \exp [ - (d\zeta/ds)(s=0) ]$ . Notice the  $\det (\mathcal{D})$  behavior of the fermion versus the  $\det (D_\mu D^\mu - m^2)^{-1}$  behavior of a complex scalar field due to the Grassmanian nature of the Gaussian path integral of the fermions. The vacuum-to-vacuum amplitude of a Majorana (real) spinor (half of the number of degrees of freedom of a complex Dirac spinor) is  $Z = \sqrt{\det (\gamma^\mu D_\mu + m)}$ . Because the complex Dirac spinor encodes both the dynamics of the electron and its anti-particle, the positron (the negative energy solutions), the vacuum-to-vacuum amplitude corresponding to the electron (positive energy solutions, propagating forward in time) must be then  $Z = \sqrt{\det (\gamma^\mu D_\mu + m)}$ .

Therefore, to sum up, the origin of the "obscure" factor  $V(D_5)^{1/4}$  in Wyler's formula is the *normalization* condition of  $V(S^4) \times V(Q_5)$  by a factor of  $V(D_5)^{1/4}$  stemming from the correspondence  $V(D_5)^{1/4} \leftrightarrow Z = \sqrt{\det (\gamma^\mu D_\mu + m)}$  and which originates from the vacuum-to-vacuum amplitude of the fermion (electron) as it propagates forward in time in the domain  $D_5$ . These last relations emerge from the correspondence between the Feynman fermion (electron) propagator in Minkowski spacetime and the Bergman Kernel of the complex homogenous domain after performing the Wyler map between an unbounded domain (the interior of the future lightcone of spacetime) to a bounded one. In general, the Bergman Kernel gives rise to a Kahler potential  $F(z, \bar{z}) = \log K(z, \bar{z})$  in terms of which the Bergman metric on  $D_n$  is given by

$$g_{i\bar{j}} = \frac{\partial^2 F}{\partial z^i \partial \bar{z}^j}. \quad (2.7)$$

We must emphasize that this Geometric probability explanation *is very different* from the interpretations provided in [3,4,5] and properly accounts for all the numerical factors. Concluding, the Geometric Probability that an electron emits a photon at  $t = -\infty$  and absorbs it at  $t = +\infty$ , is given by the *ratio* of the dimensionless measures (volumes) :

$$\begin{aligned} \alpha_{EM} &= \frac{2V(S^2)}{\Omega_{Wyler}[Q_4]} = (8\pi) \frac{1}{V(S^4)} \frac{1}{V(Q_5)} [V(D_5)]^{1/4} = \\ &= \frac{9}{8\pi^4} \left( \frac{\pi^5}{2^4 \times 5!} \right)^{1/4} = \frac{1}{137.03608\dots}. \end{aligned} \quad (2.8)$$

in very good agreement with the experimental value. This is easily verified after one inserts the values of the Euclideanized *regularized* volumes found by Hua [9]

$$V(D_5) = \frac{\pi^5}{2^4 \times 5!}, \quad V(Q_5) = \frac{8\pi^3}{3}, \quad V(S^4) = \frac{8\pi^2}{3}. \quad (2.19)$$

In general

$$V(D_n) = \frac{\pi^n}{2^{n-1} n!}, \quad V(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \quad (2.10)$$

$$V(Q_n) = V(S^{n-1} \times RP^1) = V(S^{n-1}) \times V(RP^1) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \times \pi = \frac{2\pi^{(n+2)/2}}{\Gamma(n/2)}. \quad (2.11)$$

Objections were raised to Wyler's original expression by Robertson [3]. One of them was that the hyperboloids (discs) are *not* compact and whose volumes diverge because the Lobachevsky metric diverges on the boundaries of the poly-discs. Gilmore explained [3] why one requires to use the Euclideanized regularized volumes because Wyler had shown that it is possible to map an unbounded physical domain (the interior of the future light cone) onto the interior of a homogenous bounded domain without losing the causal structure and on which there exist also a complex structure. A study of Shilov boundaries, holography and the future tube can be found in [13].

Furthermore, in order to resolve the scaling problems of Wyler's expression raised by Robertson, Gilmore showed why it is essential to use *dimensionless* volumes by setting the throat sizes of the Anti de Sitter hyperboloids to  $r = 1$ , because this is the only choice for  $r$  where all elements in the bounded domains are also coset representatives, and therefore, amount to honest group operations. Hence the so-called scaling objections against Wyler raised by Robertson were satisfactorily solved by Gilmore [3]. Thus, all the volumes in this section and in the next sections, are based on setting the scaling factor  $r = 1$ .

The question as to *why* the value of  $\alpha_{EM}$  obtained in Wyler's formula is precisely the value of  $\alpha_{EM}$  observed at the *scale* of the Bohr radius  $a_B$ , has not been solved, to our knowledge. The Bohr radius is associated with the ground (most stable) state of the Hydrogen atom [4]. The spectrum generating group of the Hydrogen atom is well known to be the conformal group  $SO(4, 2)$  due to the fact that there are two conserved vectors, the angular momentum and the Runge-Lenz vector. After quantization, one has two commuting  $SU(2)$  copies  $SO(4) = SU(2) \times SU(2)$ . Thus, it makes physical sense why the Bohr-scale should appear in this construction. Bars [15] has studied the many physical applications and relationships of many seemingly distinct models of particles, strings, branes and twistors, based on the (super) conformal groups in diverse dimensions. In particular, the relevance of two-time physics in the formulation of  $M, F, S$  theory has been advanced by Bars for some time. The Bohr radius corresponds to an energy of  $137.036 \times 2 \times 13.6 \text{ eV} \sim 3.72 \times 10^3 \text{ eV}$ . It is well known that the Rydberg scale, the Bohr radius, the Compton wavelength of electron, and the classical electron radius are all related to each other by a successive scaling in products of  $\alpha_{EM}$ .

To finalize this section and based on the MMCW  $SO(3, 2)$  Gauge Theory formulation of Gravity, with a Gauss-Bonnet topological term plus a cosmological constant, the (dimensionless) Wyler measure was *defined* as the geometric coupling strength of Gravity [4] :

$$\Omega_{Wyler}[Q_4] = \frac{V(S^4) V(Q_5)}{[V(D_5)]^{\frac{1}{4}}} \equiv \alpha_G. \quad (2.12)$$

The relationship between  $\alpha_G$  and the Newtonian gravitational  $G$  constant is based on the value of the coupling  $(1/16\pi G)$  appearing in the Einstein-Hilbert Lagrangian  $(R/16\pi G)$ , and goes as follows :

$$(16\pi G)(m_{Planck}^2) = \alpha_{EM} \alpha_G = 8\pi \Rightarrow G = \frac{1}{16\pi} \frac{8\pi}{m_{Planck}^2} = \frac{1}{2m_{Planck}^2} \Rightarrow$$



$$Gm_{proton}^2 = \frac{1}{2} \left( \frac{m_{proton}}{m_{Planck}} \right)^2 \sim 5.9 \times 10^{-39}. \quad (2.13)$$

and in natural units  $\hbar = c = 1$  yields the physical force strength of Gravity at the Planck Energy scale  $1.22 \times 10^{19} GeV$ . The Planck mass is obtained by equating the Schwarzschild radius  $2Gm_{Planck}$  to the Compton wavelength  $1/m_{Planck}$  associated with the mass; where  $m_{Planck}\sqrt{2} = 1.22 \times 10^{19} GeV$  and the proton mass is  $0.938 GeV$ . Some authors define the Planck mass by absorbing the factor of  $\sqrt{2}$  inside the definition of  $m_{Planck} = 1.22 \times 10^{19} GeV$ .

### 3 THE WEAK AND STRONG COUPLINGS

We turn now to the derivation of the other coupling constants. The Fiber Bundle picture of the previous section is essential in our construction. The Weak and the Strong geometric coupling constant strength, defined as the probability for a particle to emit and later absorb a  $SU(2)$ ,  $SU(3)$  gauge boson, respectively, can both be obtained by using the main formula derived from Geometric Probability (as ratios of dimensionless measures/volumes) after one identifies the suitable homogeneous domains and their Shilov boundaries to work with.

The physical reasoning behind the role of the Shilov boundaries is the following. It was mentioned earlier that Wyler had shown that it is possible to map an unbounded physical domain (the interior of the future light cone) onto the interior of a homogenous bounded domain without losing the causal structure and on which there exists also a complex structure. Massless gauge bosons live on the lightcone, a null boundary in Minkowski spacetime. Thus, upon performing the Wyler map, the gauge bosons are confined to live on the Shilov boundary. Because the  $SU(2)$  bosons  $W^\pm, Z^0$  and the eight  $SU(3)$  gluons have *internal* degrees of freedom (they carry weak and color charges) one must also include the measure associated with their respective internal spaces; namely, the relevant measures relevant to Geometric Probability calculations are the measures corresponding to the appropriate sphere bundles fibrations defined over the complex bounded homogenous domains  $S^m \rightarrow E \rightarrow \mathcal{D}_n$ .

Furthermore, the Geometric Probability interpretation for  $\alpha_{Weak}, \alpha_{Strong}$  agrees with Wheeler's ideas [4] that one must normalize these geometric force strengths with respect to the geometric force strength of gravity  $\alpha_G = \Omega_{Wyler}[Q_4]$  found in the last section. Hence, after these explanations, we will show below why the weak and strong couplings are given, respectively, by the *ratio* of the measures (dimensionless volumes) :

$$\alpha_{Weak} = \frac{\Omega[Q_3]}{\Omega_{Wyler}[Q_4]} = \frac{\Omega[Q_3]}{\alpha_G} = \frac{\Omega[Q_3]}{(8\pi/\alpha_{EM})}. \quad (3.1)$$

$$\alpha_{Color} = \frac{\Omega[squashed S^5]}{\Omega_{Wyler}[Q_4]} = \frac{\Omega[squashed S^5]}{\alpha_G} = \frac{\Omega[squashed S^5]}{(8\pi/\alpha_{EM})}. \quad (3.2)$$

As always, one must insert the values of the regularized (Euclideanized) dimensionless volumes provided by Hua [9] (set the scale  $r = 1$ ). We must also clarify and emphasize that we define the quantities  $\alpha_{weak}, \alpha_{color}$  as the probabilities  $\tilde{g}_w^2, \tilde{g}_c^2$ , by absorbing the

factors of  $4\pi$  in the conventional  $\alpha_w = (g_w^2/4\pi), \alpha_c = (g_c^2/4\pi)$  definitions (based on the Renormalization Group (RG) program) into our definitions of probability  $\tilde{g}_w^2, \tilde{g}_c^2$ .

Let us evaluate the  $\alpha_{Weak}$ . The internal symmetry space is  $CP^1 = SU(2)/U(1)$  ( a sphere  $S^2 \sim CP^1$ ) where the isospin group  $SU(2)$  acts via isometries on  $CP^1$ . The Shilov boundary of  $D_2$  is  $Q_2 = S^1 \times RP^1$  but is not adequate enough to accommodate the action of the isospin group  $SU(2)$ . One requires to have the Shilov boundary of  $D_3$  given by  $Q_3 = S^2 \times S^1/Z_2 = S^2 \times RP^1$  that can accommodate the action of the  $SU(2)$  group on  $S^2$ . A Fiber Bundle over  $D_3 = SO(3,2)/SO(3) \times SO(2)$  whose  $H = SO(3) \sim SU(2)$  subgroup of the isotropy group (at the origin )  $K = SO(3) \times SO(2)$  acts on  $S^2$  by simple rotations. Thus, the relevant measure is related to the fiber bundle  $E$  restricted to  $Q_3$  and is written as  $V(E|_{Q_3})$ .

One must notice that due to the fact that the  $SU(2)$  group is a double-cover of  $SO(3)$ , as one goes from the  $SO(3)$  action on  $S^2$  to the  $SU(2)$  action on  $S^2$  , one must take into account an extra factor of 2 giving then

$$V(CP^1) = V(SU(2)/U(1)) = 2 V(SO(3)/U(1)) = 2 V(S^2) = 8\pi. \quad (3.3)$$

In order to obtain the weak coupling constant due to the exchange of  $W^\pm Z^0$  bosons in the four-point tree-level processes involving four leptons, like the electron, muon, tau, and their corresponding neutrinos (leptons are fundamental particles that are lighter than mesons and baryons) which are confined to move in the interior of the domain  $D_3$ , and can emit (absorb)  $SU(2)$  gauge bosons,  $W^\pm Z^0$ , in the respective  $s, t, u$  channels, one must take into account a factor of the square root of the determinant of the fermionic propagator,  $\sqrt{\det \mathcal{D}^{-1}} = \sqrt{\det (\gamma^\mu D_\mu + m)^{-1}}$ , for each *pair* of leptons, as we did in the previous section when an electron emitted and absorbed a photon. Since there are *two* pairs of leptons in these four-point tree-level processes involving *four* leptons, one requires *two* factors of  $\sqrt{\det (\gamma^\mu D_\mu + m)^{-1}}$ , giving a net factor of  $\det (\gamma^\mu D_\mu + m)^{-1}$  and which corresponds now to a net normalization factor of  $k_n^{1/2} = (1/V(D_3))^{1/2}$ , after implementing the Feynman kernel  $\leftrightarrow$  Bergman kernel correspondence. Therefore, after taking into account the result of eq-(3-3), the measure of the  $S^2 \rightarrow E \rightarrow D_3$  bundle, restricted to the Shilov boundary  $Q_3$ , and weighted by the net normalization factor  $(1/V(D_3))^{1/2}$ , is

$$\Omega(Q^3) = 2V(S^2) \frac{V(Q_3)}{V(D_3)^{1/2}}. \quad (3.4)$$

Therefore, the Geometric probability expression is given by the ratio of measures (dimensionless volumes) :

$$\begin{aligned} \alpha_{Weak} &= \frac{\Omega[Q^3]}{\Omega_{Wyley}[Q_4]} = \frac{\Omega[Q^3]}{\alpha_G} = \frac{2V(S^2) V(Q_3)}{V(D_3)^{1/2}} \frac{\alpha_{EM}}{8\pi} = \\ &= (8\pi) (4\pi^2) \left(\frac{\pi^3}{24}\right)^{-1/2} \frac{\alpha_{EM}}{8\pi} = 0.2536.... \end{aligned} \quad (3.5)$$

that corresponds to the weak coupling constant ( $g^2/4\pi$  based on the RG convention) at an energy of the order of

$$E = M = 146 \text{ GeV} \sim \sqrt{M_{W_+}^2 + M_{W_-}^2 + M_Z^2}. \quad (3.6)$$

after we have inserted the expressions (setting the scale  $r = 1$ )

$$V(S^2) = 4\pi. \quad V(Q_3) = 4\pi^2. \quad V(D_3) = \frac{\pi^3}{24}. \quad (3.7)$$

into the formula (3-5) . The relationship to the Fermi coupling  $G_{Fermi}$  goes as follows (after setting the energy scale  $E = M = 146 \text{ GeV}$ ) :

$$G_F \equiv \frac{\alpha_W}{M^2} \Rightarrow G_F m_{proton}^2 = \left(\frac{\alpha_W}{M^2}\right) m_{proton}^2 = 0.2536 \times \left(\frac{m_{proton}}{146 \text{ GeV}}\right)^2 \sim 1.04 \times 10^{-5}. \quad (3.8)$$

in very good agreement with experimental observations. Once more, it is unknown why the value of  $\alpha_{Weak}$  obtained from Geometric Probability corresponds to the energy scale related to the  $W_+, W_-, Z_0$  boson mass, after spontaneous symmetry breaking.

Finally, we shall derive the value of  $\alpha_{Color}$  from eq-(3-2) after one defines what is the suitable fiber bundle. The calculation is based on the book by L. K. Hua [9] (pages 40, 93). The symmetric space with the  $SU(3)$  color force as a local group is  $SU(4)/SU(3) \times U(1)$  which corresponds to a bounded symmetric domain of type  $I(1, 3)$  and has a Shilov boundary that Hua calls the "characteristic manifold"  $CI(1, 3)$ . The volume  $V(CI(m, n))$  is:

$$V(CI) = \frac{(2\pi)^{mn-m(m-1)/2}}{(n-m)!(n-m+1)!\dots(n-1)!}. \quad (3.9)$$

so that for  $m = 1$  and  $n = 3$  the relevant volume is then  $V(CI) = (2\pi)^3/2! = 4\pi^3$ . We must remark at this point that  $CI(1, 3)$  is *not* the standard round  $S^5$  but is the *squashed* five-dimensional  $\tilde{S}^5$  <sup>2</sup> .

The domain of which  $CI(1, 3)$  is the Shilov boundary is denoted by Hua as  $RI(1, 3)$  and whose volume is

$$V(RI) = \frac{1!2!\dots(m-1)!1!2!\dots(n-1)! \pi^{mn}}{1!2!\dots(m+n-1)!}. \quad (3.10)$$

so that for  $m = 1$  and  $n = 3$  it gives  $V(RI) = 1!2!\pi^3/1!2!3! = \pi^3/6$  and it also agrees with the volume of the standard six-ball.

The internal symmetry space (fibers) is  $CP^2 = SU(3)/U(2)$  whose isometry group is the color  $SU(3)$  group. The base space is the 6D domain  $B_6 = SU(4)/U(3) = SU(4)/SU(3) \times U(1)$  whose subgroup  $SU(3)$  of the isotropy group (at the origin)  $K = SU(3) \times U(1)$  acts on the internal symmetry space  $CP^2$  via isometries. In this special case, the Shilov and ordinary topological boundary of  $B_6$  both coincide with the *squashed*  $S^5$  [4].

Because Gilmore, in response to Robertson's objections to Wyler's formula, has shown that one must set the scale  $r = 1$  and use *dimensionless* volumes, a bundle map  $E \rightarrow E'$  from the bundle  $CP^2 \rightarrow E \rightarrow B_6$  to the bundle  $S^4 \rightarrow E' \rightarrow B_6$  is required, because if we were to equate the volumes  $V(CP^2) = V(S^4, r = 1)$  [4], this would be tantamount of choosing another scale  $R \neq 1$  in  $CP^2$  that is *different* from the scale  $r = 1$  of  $S^4$ , as required by Gilmore. Upon performing the bundle map one can go ahead and replace the  $V(CP^2)$  for  $V(S^4, r = 1)$ .

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<sup>2</sup>Frank (Tony) Smith, private communication

Using again the same results described after eq-(2-2), since a quark can emit and absorb later on a  $SU(3)$  gluon (a one-loop process), and is confined to move in the interior of the domain  $B_6$ , there is *one* factor only of the square root of the determinant of the Dirac propagator  $\sqrt{\det \mathcal{D}^{-1}} = \sqrt{\sqrt{\det (D_\mu D^\mu - m^2)^{-1}}}$  and which is associated with a normalization factor of  $k_n^{1/4} = (1/V(B_6))^{1/4}$ . Therefore, the measure of the bundle  $S^4 \rightarrow E' \rightarrow B_6$  restricted to the *squashed*  $S^5$  (Shilov boundary of  $B^6$ ), and weighted by the normalization factor  $(1/V(B_6))^{1/4}$ , is then

$$\Omega[\textit{squashed } S^5] = \frac{V(S^4) V(\textit{squashed } S^5)}{V(B_6)^{1/4}}. \quad (3.11)$$

and the ratio of measures

$$\begin{aligned} \alpha_s &= \frac{\Omega[\textit{squashed } S^5]}{\Omega_{Wylter}[Q_4]} = \frac{\Omega[\textit{squashed } S^5]}{\alpha_G} = \frac{V(S^4) V(\textit{squashed } S^5)}{V(B_6)^{1/4}} \frac{\alpha_{EM}}{8\pi} = \\ &= \left(\frac{8\pi^2}{3}\right) (4\pi^3) \left(\frac{\pi^3}{6}\right)^{-1/4} \frac{\alpha_{EM}}{8\pi} = 0.6286\dots \end{aligned} \quad (3.12)$$

matches remarkably the strong coupling value  $\alpha_s = g^2/4\pi$  based on the one-loop Renormalization Group flow of the strong coupling :

$$\alpha_s(E^2) = \alpha_s(E_o^2) \left[ 1 + \frac{(11 - \frac{2}{3}N_f)}{4\pi} \alpha_s(E_o^2) \ln\left(\frac{E^2}{E_o^2}\right) \right]^{-1} = 0.6276 \sim 0.6286. \quad (3.13)$$

at an energy related precisely to the pion masses [3] :

$$E = 241 \text{ MeV} = 0.241 \text{ GeV} \sim \sqrt{m_{\pi^+}^2 + m_{\pi^-}^2 + m_{\pi^0}^2}. \quad (3.14)$$

and where we have taken the reference value of  $\alpha_s(E_o^2 = (34 \text{ GeV})^2) = 0.1407$  that falls within the experimental values  $\alpha_s(E_o^2 = (34 \text{ GeV})^2) = 0.14 \pm 0.02$  [27] and the number of quarks flavors is  $N_f = 6$ . The pions are the known lightest quark-antiquark pairs that feel the strong interaction. Once again, it is unknown why the value of  $\alpha_{Color}$  obtained from Geometric Probability corresponds to the energy scale related to the masses of the three pions [4] .

## 4 MERSENNE PRIMES $p$ -ADIC HIERARCHY AND OTHER APPROACHES

To conclude, we briefly mention other approaches to the determination of the physical parameters. A hierarchy of coupling constants, including the cosmological constant, based on Seifert-spheres fibrations was undertaken by [25]. The ratios of particle masses, like the proton to electron mass ratio  $m_p/m_e \sim 6\pi^5$  has also been calculated using the volumes of homogeneous bounded domains [4], [5] . A charge-mass-spin relationship was investigated in [26]. It is not known whether this procedure should work for Grand Unified Theories (GUT) based on the groups like  $SU(5)$ ,  $SO(10)$ ,  $E_6$ ,  $E_7$ ,  $E_8$ , meaning whether or not one

could obtain, for example, the  $SU(5)$  coupling constant consistent with the Grand Unification Models based on the  $SU(5)$  group and with the Renormalization Group program at the GUT scale.

Beck [7] has obtained all of the Standard Model parameters by studying the numerical minima (and zeros) of certain potentials associated with the Kaneko coupled two-dim lattices (two-dim non-linear sigma-like models which resemble Feynman's chess-board lattice models) based on Stochastic Quantization methods. The results by Smith [4] (also based on Feynman's chess board models and hyper-diamond lattices ) are analytical rather than being numerical [7] and it is not clear if there is any relationship between these latter two approaches. Noyes has proposed an iterated numerical hierarchy based on Mersenne primes  $M_p = 2^p - 1$  for *certain* values of  $p = \text{primes}$  [19], and obtained a quite large number of satisfactory values for the physical parameters. An interesting coincidence is related to the iterated Mersenne prime sequence

$$M_2 = 2^2 - 1 = 3, \quad M_3 = 2^3 - 1 = 7, \quad M_7 = 2^7 - 1 = 127, \quad 3 + 7 + 127 = 137$$

$$M_{127} = 2^{127} - 1 \sim 1.69 \times 10^{38} \sim \left(\frac{M_{Planck}}{m_{proton}}\right)^2. \quad (4.1)$$

Pitkanen has also developed methods to calculate physical masses recurring to a  $p$ -adic hierarchy of scales based on Mersenne primes [20].

An important connection between anomaly cancellation in string theory and perfect even numbers was found in [22]. These are numbers which can be written in terms of sums of its divisors, including unity, like  $6 = 1 + 2 + 3$ , and are of the form  $P(p) = \frac{1}{2}2^p(2^p - 1)$  if, and only if,  $2^p - 1$  is a Mersenne prime. Not all values of  $p = \text{prime}$  yields primes. The number  $2^{11} - 1$  is not a Mersenne prime, for example. The number of generators of the anomaly free groups  $SO(32)$ ,  $E_8 \times E_8$  of the 10-dim superstring is 496 which is an even perfect number. Another important group related to the unique tadpole-free bosonic string theory is the  $SO(2^{13}) = SO(8192)$  group related to the bosonic string compactified on the  $E_8 \times SO(16)$  lattice. The number of generators of  $SO(8192)$  is an even perfect number since  $2^{13} - 1$  is a Mersenne prime. For an introduction to  $p$ -adic numbers in Physics and String theory see [21].

A lot more work needs to be done to be able to answer the question: is all this just a mere numerical coincidence or is it design ? However, the results of the previous section indicate that it is very *unlikely* that these results were just a mere numerical coincidence (senseless numerology) and that indeed the values of the physical constants could be actually calculated from pure thought, rather than invoking the anthropic principle; i.e. namely, based on the interplay of harmonic analysis, geometry, topology, higher dimensions and, ultimately, number theory. The fact that the coupling constants involved the ratio of measures (volumes) may cast some light on the role of the world-sheet areas of strings, and world volumes of  $p$ -branes, as they propagate in target spacetime backgrounds of diverse dimensions.

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