H3 and H4 have very interesting geometry related to singularities. As Ian Porteous says in his book Geometric Differentiation for the Intelligence of Curves and Surfaces (Cambridge, 2nd ed 2001)

"... the groups Ak , Bk = Ck , Dk , E6 , E7 , E8 , F4 and G2 , had already turned up in singularity theory ...

Apart from the groups of symmetries of regular plane polygons there are only two other Coxeter groups, H3 , the full group of isometries of an icosahedron , and H4 , the full group of symmetries of an analogue of the icosahedron in R4 ...

the variety of non-regular orbits of H3 is A-equivalent to the full involute of a plane curve with an ordinary inflection. ...

Shcherbak proved ... that the full involute of the surface e of Therem 14.14 is A-equivalent to the variety of non-regular orbits of the group H4 ... the parameterization that he gives for the variety of nonregular orbits of the group H4 ...[is]...

(a,b,c) -> ( a , a c + (1/2) b^2 , a b^3 + (1/2) c^2 , a b^3 c + (1/5) b^5 + (1/3) c^3 )

a three-dimensional variety in R4 whose intersection with the hyperplane a = 0 is the two-dimensional veriety in R3 with the parameterization

(b,c) -> ( (1/2) b^2 , (1/2) c^2 , (1/5) b^5 + (1/3) c^3 )

... The H4 configuration is discussed ... in the Liverpool thesis of Alex Flegmann ... Figure... 14.9 ... taken from Flegmann's thesis ...[ published in 1985 as "Evolutes Involutes and The Coxeter Group H4", ISSN 0755-3390, Universite Louis Pasteur, Department de Mathematique, l'Institut de Recherche Mathematique Advancee, Strasbourg ]... illustrate[s] the sections of this variety in the case that a = 1 ... ".
As Porteous also says, the E8 singularity is a simple singularity whose canonical form map-germ is $x^3 + y^5$.

Flegmann's thesis also described the groups E8 and H4 as having:

Order:
192 x 10! = 696,729,600 for E8 and 14,400 for H4

No. of Reflections in each Conjugacy Class:
120 for E8 and 60 for H4

Degrees of Basic Invariants:
2, 8, 12, 14, 18, 20, 24, 30 for E8 and 2, 12, 20, 30 for H4

Note that 2, 12, 20, 30 are in both E8 and H4
and that 8, 18 are 6 greater than 2, 12
and that 14, 24 are 6 less than 20, 30
and that
\[(1 + i \sqrt{5}) (1 - i \sqrt{5}) = 1 + 5 = 6\]
is related to \[2 \times \text{Golden Ratio} = (1 + \sqrt{5})\]

Shcherbak worked with Arnol'd, and published
a 1983 paper "Singularities of a family of evolvents in the neighbourhood of a point of inflection of a curve, and the group H3 generated by reflections" (Funktsional'. Anal. i Prilozhen, 17:4, 70-2)
and
a 1984 paper "H4 in the problem of avoiding an obstacle" (Uspekhi Mat. Nauk. 39:5, 256).

Shcherbak's 1988 paper "Wavefronts and reflection groups" (Uspekhi Mat. Nauk. 43:3, 125-60) was posthumous, and was the basis for the papers by Fring and Korff at hep-th/0509152 and hep-th/0506226. John Baez, in his week 270 (11 October 2008), discussed H4 and E8 in the context of the later paper. In the earlier paper, entitled "Affine Toda field theories related to Coxeter groups of non-crystallographic type", Fring and Korff said:

"... E8 structure ...[exists]... in the form of a (minimal) E8-affine Toda field theory
... there is an even more fundamental structure than E8 underlying this particular model,
the non-crystallographic Coxeter group H4.
We draw here on the observation made first by Sherbak in 1988 ...
namely that H4 can be embedded into E8 ...
Loosely speaking, one may regard the E8-theory as two copies of H4-theories.
We get a first glimpse of this structure from a more physical point of view when we bring the mass spectrum of minimal E8-affine Toda field theory ... into the form
m1 = 1
m2 = 2 \cos(\pi/30)
m3 = \sqrt{\sin(11\pi/30) / \sin(\pi/30)}
m4 = 2 \Phi \cos(7\pi/30)
m5 = \Phi m1
m6 = \Phi m3
m7 = \Phi m3
m8 = \Phi m4
...
We observe here that there are four "fundamental" masses present in the theory, whereas the other ones can be obtained simply by a multiplication with the golden ratio

\[ \text{PHI} = \left( \frac{1}{2} \right) \left( 1 + \sqrt{5} \right) = \text{PHI}^2 - 1. \]

... Note that higher powers of PHI can be reduced to that form \( A + \text{PHI} B \) with \( A, B \) in \( \mathbb{Q} \) by a repeated use of ...

\[ \text{PHI}^2 = 1 + \text{PHI} \]
\[ \text{PHI}^3 = 1 + 2 \text{PHI} \]
\[ \text{PHI}^4 = 2 + 3 \text{PHI} \]
\[ \text{PHI}^5 = 3 + 5 \text{PHI} \]
...

\[ \text{PHI}^n = f_{(n-1)} + \text{PHI} f_n \]

where \( f_n \) is then-th Fibonacci number obeying the recursive relation

\[ f_{(n+1)} = f_n + f_{(n-1)} \]

... It will turn out that each of the sets \((m_1,m_2,m_3,m_4)\) and \((m_5,m_6,m_7,m_8)\)
can be associated with an H4-[affine Toda field theory]...

...[so that]...

H4 can be embedded into E8 ... such that the non-crystallographic structure is
"visible" inside the theories related to crystallographic Coxeter groups.

... [ There is ] a map \( w \) from a root system ... which is invariant under the action of a
crystallographic Coxeter group ...[ E8 ]... of rank ..[ 8 ]... into the union of two sets ...
related to a non-crystallographic group ...[ H4 ]... of rank ..[ 8/2 = 4 ]...
Introducing a ... labelling for the vertices on the Coxeter graphs, or equivalently the
simple roots, we can always realize this map as ...

\[ a_{E8_i} \mapsto w(a_{E8_i}) = a_{H4_i} \text{ for } i \text{ from } 1 \text{ to } 4 \]
\[ a_{E8_i} \mapsto w(a_{E8_i}) = \text{PHI} a_{H4_(i-4)} \text{ for } i \text{ from } 5 \text{ to } 8 \]

...
The corresponding Cartan matrix of $E_8$ together with its construction from inner products in $\tilde{\Delta}$ in agreement with (2.3) is

$$K = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} = R \left( \begin{pmatrix} \tilde{K} & \phi \tilde{K} \\ \phi \tilde{K} & \phi^2 \tilde{K} \end{pmatrix} \right). \quad (2.38)$$

The Cartan matrix of $H_4$ reads

$$\tilde{K} = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -\phi \\
0 & 0 & -\phi & 2
\end{pmatrix}. \quad (2.39)$$

... the map $w$ is an isometric isomorphism, such that we may compute inner products in the root system $[E_8]$ from inner products in $[H_4]$

$$A \cdot B = R( w(A) \cdot w(B) )$$

Here the map $R$, called a rational form relative to $\Phi$, extracts from a number of the form $A + \Phi B$ with $A, B$ in $Q$ the rational part of $A$

$$R(A + \Phi B) = A$$

We normalize all our roots to have length 2, we may therefore compute the Cartan matrix related to $[E_8]$ entirely from inner products in $[H_4]$ [and]... compute inner products in $[H_4]$ from those in $[E_8]$...

... the entire root system $[E_8]$... can be separated into $[8]$... orbits $W_i$... each containing $[30]$... roots... as shown in this image from a video by Garrett Lisi for FQXi07 on his web site deferentialgeometry.org
The vertices ... of the Coxeter graph ... separate into two disjoint sets $V^+$ and $V^-$.

The Coxeter numbers ... [for E8 and H4] are 30 and [with respect to the map $w$] the set of exponents separate ... into

$$\{1, 7, 11, 13, 17, 19, 23, 29\} = \{1, 11, 19, 29\} \cup \{7, 13, 17, 23\}$$

Note that 1, 11, 19, 29 are in both E8 and H4 and that 7, 17 are 6 greater than 1, 11.
and that 13, 23 are 6 less than 19, 29
and that
\[(1 + i \sqrt{5}) (1 - i \sqrt{5}) = 1 + 5 = 6\] is related to \(2 \times \text{Golden Ratio} = (1 + \sqrt{5})\) … 

E8 polytope has 240 vertices,
the sum of two H4 polytopes, each with 120 vertices, related by Golden Ratio,
so E8 contains two copies of H4 related by Golden Ratio

In more detail:

\[248 - \text{dim E8} = 120 - \text{dim D8} + 128 - \text{dim half-spinorD8}\]

\[240 \text{ E8 root vector vertices} = 112 \text{ D8 root vector vertices} + 128 \text{ half-spinor vertices}\]

D8 contains two copies of D4
Each D4 has 24 root vector vertices forming a 24-cell

Golden Ratio points can be chosen (in two different ways) on each of the 96 edges of a 24-cell so that the 96 Golden Ratio points plus the 24 vertices form the \(96 + 24 = 120\) vertices of a 600-cell.

So, the two D4 of D8 give two 600-cells.
Each 600-cell acts as a 120-vertex symmetry polytope for a copy of H4.

In E8 physics, 8-dimensional spacetime is, at our low energies, reduced to
4-dimensional physical spacetime (denoted here 4)
plus
4-dimensional internal symmetry space (denoted here by 4*).

Each of the two D4 have 4-dimensional Cartan subalgebra spaces:
The Cartan space of one D4, denoted by D4, corresponds to 4-dim physical spacetime, denoted by 4. That D4 produces Gravity.
The Cartan space of the other D4, denoted by D4*, corresponds to 4-dim internal symmetry space, denoted by 4*. That D4 produces the Standard Model.
E8 = D8 + half-spinorD8 = \((D4 + D4^* + 64) + (64 + 64)\)

The 64 = 8x8 in \((D4 + D4^* + 64)\) represents 8 Dirac gamma components of the dimensions of 8-dim spacetime that is reduced to 4 + 4*.
E8 physics dimensional reduction of 8-dim spacetime to 4 + 4* also reduces the 8 Dirac gammas to 4 + 4*
so that 64 = 8x8 = \((4 + 4^*) \times 8 \times 4 + (4 + 4^*) \times 4^* = 32 + 32^*\)
so that the 32 corresponds to the D4 of physical spacetime
and the 32* corresponds to the D4* of internal symmetry space.

Note that the 32 of D4 has some connection to internal symmetry space, as might be expected from the detailed structure of the SU(3) sector of Bataakis M4\times CP2
Kaluza-Klein theory, in that the 32 includes the 4 physical spacetime Dirac gamma components of the four dimensions of 4* internal symmetry space.

Each of the 64 = 8x8 in \((64 + 64)\) represents 8 Dirac gamma components of either 8 fermion particles or 8 fermion antiparticles,
so each of those 64 = 8x8 = 8(4+4*) = 32 + 32*
where the 32 represents components with respect to 4-dim physical spacetime
and the 32* represents components with respect to 4-dim internal symmetry space
and
E8 = D8 + half-spinorD8 = \((D4 + D4^* + 32 + 32^*) + (32 + 32^* + 32 + 32^*)\)

E8 = \([ (D4 + 32) + (32 + 32) ] + [ (D4^* + 32^*) + (32^* + 32^*) ]\)

E8 = \([ 24 + 96 ] + [ 24^* + 96^* ] = 120 + 120^* = H4 + H4^*\)

where
H4 is a copy of H4 that corresponds to D4, Gravity, and 4-dim physical spacetime
and
H4* is a copy of H4 that corresponds to D4*, the Standard Model, and 4-dim internal symmetry space.

The following 4 pages from Appendix B of hep-th/0506226 by Fring and Korff show “… explicit computations of various orbits of … roots related to non-crystallographic and crystallographic Coxeter groups and exhibit how they can be embedded into one another …”: 
B. The orbits of $H_4$ and $E_8$

Successive action of $\sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_7 \sigma_2 \sigma_6 \sigma_4 \sigma_8$ and $\tilde{\sigma} = \tilde{\sigma}_3 \tilde{\sigma}_2 \tilde{\sigma}_4$ yields

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\Omega_1$</th>
<th>$\omega(\Omega_1) = \tilde{\Omega}_1$</th>
<th>$\Omega_5$</th>
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<tbody>
<tr>
<td>$\sigma^0$</td>
<td>$\alpha_1$</td>
<td>$\tilde{\alpha}_1$</td>
<td>$\alpha_5$</td>
</tr>
<tr>
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<td>$\alpha_6 + \alpha_7$</td>
</tr>
<tr>
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</tr>
<tr>
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<td>$\alpha_3 + \alpha_6 + \alpha_7 + \alpha_8$</td>
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<tr>
<td>$\sigma^8$</td>
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<td>$\phi (\tilde{\alpha}_1 + \tilde{\alpha}_2) + \phi^2 (\tilde{\alpha}_3 + \tilde{\alpha}_4)$</td>
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<td>( \Omega_6 )</td>
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<td>( \tilde{\alpha}_2 )</td>
<td>( -\alpha_6 )</td>
</tr>
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<td>( a_5 + a_6 + a_7 )</td>
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<td>( \phi^2(\tilde{a}_1 + 2\tilde{a}_3) + \phi^3(\tilde{a}_2 + \tilde{a}_4) )</td>
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<td>( a_1 + a_2 + 2a_3 + 2a_4 ) + ( a_5 + 2a_6 + 3a_7 + 3a_8 )</td>
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<td>( \phi \tilde{a}_2 + \phi^2 (\tilde{a}_3 + \tilde{a}_4) )</td>
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<td>( \tilde{a}_2 )</td>
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<td>(\Omega_7)</td>
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<td>(\sigma^0)</td>
<td>(a_3)</td>
<td>(\tilde{a}_3)</td>
<td>(a_7)</td>
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<td>(\tilde{a}_1 + \tilde{a}_2 + \phi^2 \tilde{a}_3 + \phi \tilde{a}_4)</td>
<td>(a_3 + a_4 + a_5 + a_6 + 2a_7 + a_8)</td>
</tr>
<tr>
<td>(\sigma^2)</td>
<td>(a_2 + a_3 + a_4 + a_5)</td>
<td>(a_6 + 2a_7 + a_8)</td>
<td>(\phi \tilde{a}_1 + \phi^2 \tilde{a}_2 + \phi^3 \tilde{a}_3 + \phi^4 \tilde{a}_4)</td>
</tr>
<tr>
<td>(\sigma^3)</td>
<td>(a_3 + a_4 + a_5)</td>
<td>(+ 3a_7 + 2(a_6 + a_8))</td>
<td>(\phi(\tilde{a}_1 + 2\tilde{a}_2 + (\phi^4 - 1) \tilde{a}_3 + \phi^3 \tilde{a}_4))</td>
</tr>
<tr>
<td>(\sigma^4)</td>
<td>(a_1 + a_2 + 2(a_3 + a_4))</td>
<td>(+ a_5 + 3a_7 + 2(a_6 + a_8))</td>
<td>(\phi^2(\tilde{a}_1 + \phi \tilde{a}_2 + \phi^2 \tilde{a}_3 + 2\tilde{a}_4))</td>
</tr>
<tr>
<td>(\sigma^5)</td>
<td>(a_1 + a_4 + 2(a_2 + a_6))</td>
<td>(+ a_5 + 3(a_3 + a_7 + a_8))</td>
<td>(\phi^2(\tilde{a}_1 + 2\tilde{a}_2 + 3\tilde{a}_3) + (\phi^4 - 1) \tilde{a}_4)</td>
</tr>
<tr>
<td>(\sigma^6)</td>
<td>(2(a_2 + a_3 + a_4 + a_6))</td>
<td>(+ a_5 + 3(a_3 + a_7 + a_8))</td>
<td>(\phi^2(\tilde{a}_1 + 2\tilde{a}_2 + 2\phi \tilde{a}_3 + \phi^2 \tilde{a}_4))</td>
</tr>
<tr>
<td>(\sigma^7)</td>
<td>(a_2 + 2(a_3 + a_4 + a_5))</td>
<td>(+ a_5 + 3(a_3 + a_7 + a_8))</td>
<td>(\phi(\tilde{a}_1 + 2\tilde{a}_2 + 3\tilde{a}_3 + \phi^2 \tilde{a}_4))</td>
</tr>
<tr>
<td>(\sigma^8)</td>
<td>(a_1 + a_2 + 2(a_3 + a_4))</td>
<td>(+ a_5 + 4a_7 + 3(a_6 + a_8))</td>
<td>(\phi^2(\tilde{a}_1 + 2\phi \tilde{a}_3 + \phi^2 \tilde{a}_4) + (\phi^4 - 1) \tilde{a}_2)</td>
</tr>
<tr>
<td>(\sigma^9)</td>
<td>(a_1 + 2(a_2 + a_4 + a_6))</td>
<td>(+ a_5 + 3(a_3 + a_7 + a_8))</td>
<td>(\phi^2(\tilde{a}_1 + 2\tilde{a}_2 + 3\tilde{a}_3 + \phi^3 \tilde{a}_4))</td>
</tr>
<tr>
<td>(\sigma^{10})</td>
<td>(a_1 + 2(a_2 + a_3 + a_6))</td>
<td>(+ a_4 + a_5 + 3(a_7 + a_8))</td>
<td>(\phi^2(\tilde{a}_1 + 2\tilde{a}_2 + 3\tilde{a}_3) + \phi \tilde{a}_4)</td>
</tr>
<tr>
<td>(\sigma^{11})</td>
<td>(a_2 + a_3 + a_5 + 3a_7)</td>
<td>(+ 2(a_4 + a_6 + a_8))</td>
<td>(\phi \tilde{a}_1 + \phi^3 \tilde{a}_2 + (\phi^4 - 1) \tilde{a}_3 + 2\phi \tilde{a}_4)</td>
</tr>
<tr>
<td>(\sigma^{12})</td>
<td>(a_3 + a_4 + a_5)</td>
<td>(+ 2(a_6 + a_7 + a_8))</td>
<td>(\phi \tilde{a}_1 + 2\phi \tilde{a}_2 + \phi^3 (\tilde{a}_3 + \tilde{a}_4))</td>
</tr>
<tr>
<td>(\sigma^{13})</td>
<td>(a_1 + a_2 + a_3 + a_4)</td>
<td>(+ a_5 + a_7 + a_8)</td>
<td>(\tilde{a}_1 + \phi^2 (\tilde{a}_2 + \tilde{a}_3 + \tilde{a}_4))</td>
</tr>
<tr>
<td>(\sigma^{14})</td>
<td>(a_2 + a_3 + a_8)</td>
<td>(\tilde{a}_2 + \tilde{a}_3 + \phi \tilde{a}_4)</td>
<td>(a_4 + a_6 + a_7 + a_8)</td>
</tr>
<tr>
<td>(\sigma^{15})</td>
<td>(- a_3)</td>
<td>(- \tilde{a}_3)</td>
<td>(- a_7)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\Omega_4$</td>
<td>$\omega(\Omega_4) = \Omega_4$</td>
<td>$\Omega_8$</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>-----------------</td>
<td>-------</td>
</tr>
<tr>
<td>$\sigma^0$</td>
<td>$-\alpha_4$</td>
<td>$-\tilde{\alpha}_4$</td>
<td>$-\alpha_8$</td>
</tr>
<tr>
<td>$\sigma^1$</td>
<td>$\alpha_4 + \alpha_7$</td>
<td>$\phi \tilde{\alpha}_3 + \tilde{\alpha}_4$</td>
<td>$\alpha_3 + \alpha_7 + \alpha_8$</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>$\alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$</td>
<td>$\phi (\tilde{\alpha}_1 + \tilde{\alpha}_2 + \phi \tilde{\alpha}_3 + \tilde{\alpha}_4)$</td>
<td>$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + 2 \alpha_7 + \alpha_8$</td>
</tr>
<tr>
<td>$\sigma^3$</td>
<td>$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + 2 \alpha_7 + \alpha_8$</td>
<td>$\phi \tilde{\alpha}_1 + \phi^2 (\tilde{\alpha}_2 + \phi \tilde{\alpha}_3)$</td>
<td>$\alpha_2 + \alpha_4 + \alpha_5 + 3 \alpha_7 + \alpha_8$</td>
</tr>
<tr>
<td>$\sigma^4$</td>
<td>$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + 2 \alpha_7 + \alpha_8$</td>
<td>$\phi \tilde{\alpha}_1 + \phi^2 (\tilde{\alpha}_2 + \alpha_3 + \phi \tilde{\alpha}_4)$</td>
<td>$\alpha_1 + \alpha_2 + (\alpha_3 + \alpha_4 + \alpha_6)$ + $\alpha_5 + 4 \alpha_7 + 3 \alpha_8$</td>
</tr>
<tr>
<td>$\sigma^5$</td>
<td>$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + 3 \alpha_7 + 2 (\alpha_6 + \alpha_8)$</td>
<td>$\phi \tilde{\alpha}_1 + \phi \tilde{\alpha}_2 + \phi \tilde{\alpha}_3 + \phi \tilde{\alpha}_4$</td>
<td>$\alpha_1 + 2 (\alpha_2 + \alpha_4 + \alpha_5)$ + $3 (\alpha_3 + \alpha_6 + \alpha_8) + 4 \alpha_7$</td>
</tr>
<tr>
<td>$\sigma^6$</td>
<td>$\alpha_2 + 2 (\alpha_3 + \alpha_4) + \alpha_5 + 3 \alpha_7 + 2 (\alpha_6 + \alpha_8)$</td>
<td>$\phi (\tilde{\alpha}_1 + \phi \tilde{\alpha}_2 + \phi \tilde{\alpha}_3 + \phi \tilde{\alpha}_4) + (\phi^4 - 1) \tilde{\alpha}_3$</td>
<td>$\alpha_1 + 2 (\alpha_2 + \alpha_4 + \alpha_5) + 3 (\alpha_3 + \alpha_6 + \alpha_8) + 5 \alpha_7 + 4 \alpha_8$</td>
</tr>
<tr>
<td>$\sigma^7$</td>
<td>$\alpha_1 + \alpha_2 + 2 (\alpha_3 + \alpha_6) + \alpha_4 + \alpha_5 + 3 (\alpha_7 + \alpha_8)$</td>
<td>$\phi^2 (\tilde{\alpha}_1 + \phi \tilde{\alpha}_2 + \phi^2 \tilde{\alpha}_3)$ + $(\phi^4 - 1) \tilde{\alpha}_4$</td>
<td>$\alpha_1 + 2 (\alpha_2 + \alpha_3 + \alpha_5) + 5 \alpha_7 + 4 \alpha_8$</td>
</tr>
<tr>
<td>$\sigma^8$</td>
<td>$\alpha_1 + \alpha_2 + 2 (\alpha_3 + \alpha_4) + \alpha_5 + 2 (\alpha_6 + \alpha_8) + 3 \alpha_7$</td>
<td>$\phi^2 (\tilde{\alpha}_1 + 2 \phi \tilde{\alpha}_2 + \phi^2 \tilde{\alpha}_3 + 2 \phi \tilde{\alpha}_4)$</td>
<td>$\alpha_1 + 2 (\alpha_2 + \alpha_4 + \alpha_5)$ + $3 \alpha_3 + 5 \alpha_7 + 4 (\alpha_6 + \alpha_8)$</td>
</tr>
<tr>
<td>$\sigma^9$</td>
<td>$\alpha_2 + \alpha_4 + \alpha_5 + 2 (\alpha_6 + \alpha_8) + 3 \alpha_7$</td>
<td>$\phi \alpha_1 + \phi^2 \alpha_2 + \phi^4 \alpha_3$ + $(\phi^4 - 1) \alpha_4$</td>
<td>$\alpha_1 + 2 (\alpha_2 + \alpha_3 + \alpha_5 + 5 \alpha_7)$ + $3 (\alpha_3 + \alpha_4 + \alpha_6 + 4 \alpha_8)$</td>
</tr>
<tr>
<td>$\sigma^{10}$</td>
<td>$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + 2 (\alpha_4 + \alpha_6 + \alpha_8) + 3 \alpha_7$</td>
<td>$\phi \tilde{\alpha}_2 + \phi^2 (\alpha_2 + \alpha_4)$ + $(\phi^4 - 1) \tilde{\alpha}_3$</td>
<td>$\alpha_1 + 2 (\alpha_2 + \alpha_4 + \alpha_5)$ + $3 (\alpha_3 + \alpha_6) + 4 (\alpha_7 + \alpha_8)$</td>
</tr>
<tr>
<td>$\sigma^{11}$</td>
<td>$\alpha_2 + \alpha_4 + \alpha_5 + 2 (\alpha_3 + \alpha_6 + \alpha_7 + \alpha_8)$</td>
<td>$\phi \alpha_1 + \phi^3 (\alpha_2 + \alpha_4)$ + $2 \phi^2 \alpha_3$</td>
<td>$\alpha_1 + 2 (\alpha_2 + \alpha_3 + \alpha_4)$ + $3 \alpha_3 + 4 \alpha_7 + 3 (\alpha_6 + \alpha_8)$</td>
</tr>
<tr>
<td>$\sigma^{12}$</td>
<td>$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + 2 (\alpha_7 + \alpha_8)$</td>
<td>$\tilde{\alpha}_1 + \phi^2 \tilde{\alpha}_2 + \phi^3 \tilde{\alpha}_3$ + $(\alpha_3 + \alpha_4)$</td>
<td>$\alpha_2 + 2 (\alpha_2 + \alpha_4 + \alpha_5) + \alpha_5 + 3 (\alpha_7 + \alpha_8)$</td>
</tr>
<tr>
<td>$\sigma^{13}$</td>
<td>$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + 2 (\alpha_6 + \alpha_7 + \alpha_8)$</td>
<td>$\phi \alpha_1 + \phi^2 (\alpha_2 + \alpha_3 + \alpha_4)$</td>
<td>$\alpha_1 + \alpha_2 + \alpha_3 + 4 \alpha_4 + 2 (\alpha_6 + \alpha_7 + \alpha_8)$</td>
</tr>
<tr>
<td>$\sigma^{14}$</td>
<td>$\alpha_6 + \alpha_7 + \alpha_8$</td>
<td>$\phi (\alpha_2 + \alpha_3 + \alpha_4)$</td>
<td>$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8$</td>
</tr>
<tr>
<td>$\sigma^{15}$</td>
<td>$\alpha_4$</td>
<td>$\tilde{\alpha}_4$</td>
<td>$\alpha_8$</td>
</tr>
</tbody>
</table>
The projection shown above from Garrett Lisi’s FQXi07 video of the 240 E8 root vectors from 8-dim to 2-dim with the form of 8 circles of 30 vertices each is described by H. S. M. Coxeter, in section 3.8 of his paper "Regular and Semi-Regular Polytopes III" (Math. Zeit. 200 (1988) 3-45, reprinted in "Kaleidoscopes: Selected Writings of H. S. M. Coxeter" (Wiley 1995)).

“…Du Val ... discovered ... ten-dimensional coordinates ...[ u_1 ... u_10 ]... for the ... lattice 5_21 ... In fact, the vertices of a 5_21 of edge 5 sqrt(2) are all the points in Euclidean 10-space whose coordinates satisfy the equations

\[ x_1 + x_2 + x_3 + x_4 + x_5 = x_6 + x_7 + x_8 + x_9 + x_{10} = 0 \]

and the congruences

\[ x_1 = x_2 = x_3 = x_4 = x_5 = 2 \ x_6 = 2 \ x_7 = 2 \ x_8 = 2 \ x_9 = 2 \ x_{10} \ (\text{mod} \ 5) \]

In this lattice, the points at distance 5 sqrt(2) from the origin are, of course, the 240 vertices of a 4_21 . . . In the accompanying table ...[of]... new coordinates

\[ u_v = ( x_v + t \ x_{(v+5)})/\sqrt{5} \] ...[for]... \( v = 1,2,3,4,5 \)

\[ u_v = ( t \ x_{(v-5)} - x_v )/\sqrt{5} \] ...[for]... \( v = 6,7,8,9,10 \)

where \( t = (1/2)(\sqrt{5} + 1) \) ...
... By picking out alternate rows of the right-hand column of the table, we distinguish two sets of 120 vertices of 4

one set satisfying \( u_1^2 + \ldots + u_5^2 = 10 \) [and] \( u_6^2 + \ldots u_{10}^2 = 10 t^2 \)

and the other satisfying \( u_1^2 + \ldots + u_5^2 = 10 t^2 \) [and] \( u_6^2 + \ldots u_{10}^2 = 10 \)

Let us call these 'odd' and 'even' vertices, respectively. In Fig. 3.8 d …

… they appear as black and white dots. ... When we project onto the 5-space \( u_6 = \ldots = u_{10} = 0 \) by ignoring the last five coordinates, we obtain the 120 + 120 vertices of two homothetic 600-cells ... one having the coordinates …[of]… the other … multiplied by \( t \)
These 240 points are the vertices of the 8-dimensional uniform polytope 4_21 …[they]… represent the 240 lattice points at distance 2 from the origin: the 16 permutations of ( +/-2,0,0,0,0,0,0,0) and the 112 + 112 cyclic permutations of (the last 7 coordinates in)

(+/−1 ; 0,0,0,+/−1,+/−1,0,+/−1) (0 ;+/−1,+/−1,+/−1,0,0,+/−1,0)

"..." using octonionic basis { 1, i, j, k, e, ie, je, ke }.

Notice that the structure can be seen as

16 + 112 + 112 = 128 + 112 = half-spinor D8 + adjoint root vectors D8.

The E8 = 120 + 120 = (4+4)x30 = 8x30 decomposition does not directly correspond to the E8 = 112 + 128 decomposion because:

Each set of 120 vertices of each of the two 600-cells is made up of

120 = 24 vertices of a D4 + 3x32 where the three 32 are related by triality so that each 600-cell contains one of the two D4 in D8, whose 112 vertices are D8 = D4 + D4 + 8x8 = 24 + 24 + 64 = 112;

and, further, you cannot put all 128 of the D8 half-spinor into the 120 vertices of one 600-cell.

In his FQXi07 video, Garrett Lisi showed that the 8-Circle E8 projection can be rotated to see the 120 + 120 = 112 + 128 = 240 root vectors of E8 from another perspective, which I will call the H4+H4 Square projection. I have put an m4v movie of the relevant part of Garrett Lisi’s video on the web at

[tony5m17h.net/8x30circletosquare.m4v](tony5m17h.net/8x30circletosquare.m4v)

so that you can see how the rotation transition works.

In the H4+H4 Square projection the 240 vertices are color-coded:
64 red for 8 components of 8 fermion particles, 
60 of which are in the outer 120 of the 240 
and 4 of which are in the inner 120 near the center; 
64 green for 8 components of 8 fermion antiparticles, 
60 of which are in the outer 120 of the 240 
and 4 of which are in the inner 120 near the center; 
64 blue for 8 components of 8 Kaluza-Klein spacetime dimensions 
in the inner 120 – 4 – 4 = 112 of D8 in E8; 
24 bright yellow for a D4 producing MacDowell-Mansouri Conformal Gravity; 
24 orange for a D4* producing the Standard Model gauge bosons. 
The 64 blue plus 24 bright yellow D4 plus 24 orange D4* make up the 
64+24+24 = 112 root vectors of the D8 in E8:
The 64 red plus 64 green make up the $64+64 = 128$ root vectors that correspond to the +half-spinors of the D8 in E8, and to the 128-dim space $E8 / D8$ of Boris Rosenfeld’s rank-8 octo-octonionic projective plane (OxO)P2.

Note that E8 contains +half-spinors of D8 representing one generation of fermions, and that E8 does NOT contain the –half-spinors of D8 representing one ANTI-generation of fermions, which enables (along with other natural math structures) my E8 physics model to be a realistic chiral model.
The H4+H4 Square projection has been used by Bathsheba Grossman to make an 8 cm glass model (see www.bathsheba.com/crystal/e8/) of the E8 root vector system:

As Bathsheba Grossman says at www.bathsheba.com/crystal/process/
“… The points are tiny (.1mm) fractures created by a focused laser beam. The conical beam, with a focal length of about 3", shines into the glass without damaging it except at the focal point. At that one point, concentrated energy heats the glass to the cracking point, causing a microfracture. To draw more points, the laser is pulsed on and off. To make the beam move between points, it's reflected from a mirror that is repositioned between pulses. The mirror is moved by computer-controlled motors, so many points can be drawn with great speed and accuracy. A typical design might use several hundred thousand points, or half a million isn't unusual in a large block, each placed with .001" accuracy. … I'm currently using … a Nd:YAG laser …”.

In an animation at www.bathsheba.com/crystal/e8/ rotation of the Bathsheba E8
Glass shows that the H4+H4 Square projection can be transformed into an E6 Hexagon projection.
Garrett Lisi said at FQXi07 (July 2007)
(see deferentialgeometry.org/talks/FQXi07/FQXi2007text.txt)
that he is
“…pretty sure … this hexagonal pattern … relates to E6 as a subgroup of E8. …”.

To see how the E6 in E8 works,
look at the \(240 = 72 + 168\) decomposition that is natural in 9-dimensional coordinates,
described by Coxeter in "Integral Cayley Numbers"
(Duke Math. J. vol. 13 no. 4, Dec. 1946,
reprinted in Coxeter's book "The Beauty of Geometry - Twelve Essays")
where Coxeter says (I am changing “l” to “z” for typographical reasons):

“…In terms of…

\[
\begin{align*}
z_1 &= (1/2)(1 + e) \\
z_2 &= (1/2)(1 - e) \\
z_3 &= (1/2)(i + ie) \\
z_4 &= (1/2)(i - ie) \\
z_5 &= (1/2)(j + je) \\
z_6 &= (1/2)(j - je) \\
z_7 &= (1/2)(k + ke) \\
z_8 &= (1/2)(k - ke) \\
\end{align*}
\]

…
the 240 vertices of \(4_{21}\) … consist of
112 like \(+/- z_1 +/- z_2\)
and
128 like \((1/2)( - z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 )\)
(with an odd number of minus signs)

…
[The 112 correspond to the 112 root vectors of D8,
and the 128 to the -half-spinors of D8.
Changing basis by changing sign of each of the basis elements
\(\{ 1, i, j, k, e, ie, je, ke \}\) gives 8 different lattices, 7 of which are independent.
The 128 with even number of minus signs that are not in E8 correspond to the
+half-spinors of D8 (physically, to an anti-generation of fermions).]
…
A … convenient notation, for some purposes, is obtained by defining …
\[ z_0 = \frac{1}{2}(1+i+j+k) = \frac{1}{2}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8) \]
… perhaps the greatest advantage of this notation is that enables us to replace the nine Cayley numbers …[ corresponding to the 9 vertices of the extended E8 Dynkin diagram of the E8 lattice 5_21 ]… by

\[ z_8 - z_7 \\
z_7 - z_6 \\
z_6 - z_5 \\
z_5 - z_4 \\
z_4 - z_3 \\
z_3 - z_2 \\
z_2 - z_1 \\
z_1 - z_0 \\
1 \]
… the 240 integral Cayley numbers of norm 1 consist of …
72 like … \( z_r - z_s \)
… and 168 like … \(+/- (z_r + z_s + z_t - z_0)\)
where \( r,s,t \) take any three distinct values among 0,1,2,3,4,5,6,7,8
…”.

The decomposition 240 = 72 + 168
is related to the 72 root vectors of the 78-dimensional E6 subgroup of 248-dimensional E8.

The E6 Hexagon projection can as shown in Garrett Lisi’s video for FQXi07 also be seen as a transformation of the 8-Circle E8 projection
I have put an m4v movie of the relevant part of Garrett Lisi’s video on the web at tony5m17h.net/8x30circletohex.m4v showing the transformation from the 8-Circle projection to the E6 Hexagon projection. Here is a more detailed image of the E6 Hexagon projection, adapted from Garrett Lisi’s video:

In the E6 Hexagon projection the 240 vertices are color-coded:
64 red for 8 components of 8 fermion particles;
64 green for 8 components of 8 fermion antiparticles;
64 blue for 8 components of 8 Kaluza-Klein spacetime dimensions;
24 bright yellow for a D4 producing MacDowell-Mansouri Conformal Gravity;
24 orange for a D4* producing the Standard Model gauge bosons.
The 72 vertices that are E6 root vectors are

![Image of E6 root vectors]

The E6 vertices are:
16 red for 2 components (\{1,i\} complex) of 8 fermion particles;
16 green for 2 components (\{1,i\} complex) of 8 fermion antiparticles;
16 blue for 2 components (\{1,i\} complex of 8-dimensional Kaluza-Klein spacetime;
24 bright yellow for a D4 producing MacDowell-Mansouri Conformal Gravity.

The 168 vertices in E8 outside E6 are:
48 red for 6 (\{i,j,k,ie,je,ke\} octonion) components of 8 fermion particles;
48 green for 6 (\{i,j,k,ie,je,ke\} octonion) components of 8 fermion antiparticles;
48 blue for 6 (\{i,j,k,ie,je,ke\} octonion) components of 8-dimensional K-K spacetime;
24 = 8+8+8 orange for a D4* producing the Standard Model gauge bosons.

The 48 red plus 8 orange give a red 48+8 = 56-dimensional Freudenthal algebra Fr(3,O).
The 48 green plus 8 orange give a green 48+8 = 56-dimensional Fr(3,O).
The 48 blue plus 8 orange give a blue 48+8 = 56-dimensional Fr(3,O).

E6 is the automorphism group of each of the three 56-dimensional Fr(3,O) and the three 56-dimensional Fr(3,O) are related by Triality.
The $168 = 56 + 56 + 56$ vertices of the three Fr(3,O) Freudenthal algebras are
The E6 itself corresponds to my E6 Bosonic Strings-as-World-Lines physics model with fermionic structure coming from orbifolding—see CERN CDS EXT-2004-031.

If you regard the orange 24 vertices, not as root vectors of second D4, but as the \(8+8+8 = 8v\) vectors + 8s+ +half-spinors + 8s- –half-spinors of the D4 in E6, then the 8v and 8s+ and 8 s- are related by triality. Since the 48 blue and 48 red and 48 green are also related to vector spacetime, +half-spinor fermion particles, and –half-spinor fermion antiparticles, they can be considered as 48v and 48s+ and 48s- which are also related by triality. Combining the 8v and 8s+ and 8s- with the 48v and 48s+ and 48s- gives 56v and 56s+ and 56s- which are related by triality, so that

\[
\text{the } 168 = 56v + 56s+ + 56s-
\]

where the three 56 are related by the triality of the D4 in E6

and
each 56 corresponds to a Freudenthal algebra Fr(3,O)
of which E6 is the automorphism group.

The 56-dimensional Freudenthal algebra Fr(3,O) is 2x2 Zorn-type vector-matrices

\[
\begin{pmatrix}
a & X \\
Y & b
\end{pmatrix}
\]

where a and b are real numbers and X and Y are elements of the 27-dimensional Jordan Algebra J(3,O) of 3x3 Hermitian Octonionic matrices

\[
\begin{pmatrix}
d & S+ & V \\
S+* & e & S- \\
V* & S-* & f
\end{pmatrix}
\]

where d, e, and f are real numbers; S+, V, and S- are Octonions; and * denotes conjugation.
Fr(3,O) includes a complexification of J(3,O), so that each Half-Spinor Fermion Representation Space has 8 Complex Dimensions and a corresponding Bounded Complex Domain with 8-real-dimensional Shilov Boundary $S^7 \times \text{RP}^1$, as does the Vector SpaceTime representation space.

Introduction of a preferred Quaternionic structure at low energies gives the Vector SpaceTime an $M_4 \times \text{CP}^2$ 8-dim Kaluza-Klein structure and a Higgs mechanism by the work of Meinhard Mayer, and produces the second and third generations of Fermion Particles and AntiParticles.

Each of the triality-related 56v and 56s+ and 56s- Freudenthal Fr(3,O) algebras has the structure of a 2x2 Zorn matrix as described by Boris Rosenfeld in his book “Geometry of Lie Groups” (Kluwer 1997) (see particularly pages 56ff and 91ff):

\[
\begin{bmatrix}
1 & 8 & 8 \\
1 & * & 1 & 8 \\
* & * & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 8 & 8 \\
* & 1 & 8 \\
* & * & 1
\end{bmatrix}
\]

The physical interpretation with respect to the E6 in E8 root vector decomposition is similar for all three of the triality-related 56v and 56s+ and 56s- Freudenthal Fr(3,O) algebras, so the following description of physical interpretation will be in terms of one of them, the 56s+ red vertices corresponding to 48 of the $64 = 8 \times 8$ components in 8 dimensions of the 8 fundamental first-generation fermion particles.

The descriptions of the 56v blue root vectors corresponding to 48 of the $64 = 8 \times 8$ components in 8 dimensions of the 8 Dirac gammas and the 56s- green vertices corresponding to 48 of the $64 = 8 \times 8$ components in 8 dimensions of the 8 fundamental first-generation fermion antiparticles are, as indicated by triality, similar.

In the Zorn matrix for the 56s+ red vertices corresponding to 48 of the $64 = 8 \times 8$ components in 8 dimensions of the 8 fundamental first-generation fermion particles, the 6 entries labeled 8 are 6 octonions, corresponding to $6 \times 8 = 48$ root vectors,
representing the \{ i, j, k, ie, je, ke \} components of each of the 8 fermion particles.

When combined with the 2x8 = 16 root vectors inside E6 representing the \{ 1, e \} components of each of the 8 fermion particles the result is 6x8 + 2x8 = 48 + 16 = 64 = 8x8 root vectors representing all 8 \{ i, j, k, ie, je, ke, e \} components, in 8 dimensions, of each of all 8 fundamental first-generation fermion particles, which correspond to 64 of the 128 D8 half-spinor root vectors of the 240 of E8.

As to the remaining Zorn matrix real-number entries labeled 1, corresponding to 8 of the 24 = 8+8+8 orange vertices outside the E6 part of E8

\[
\begin{array}{cccc}
1 & & & \\
1 & 1 & & \\
& 1 & & \\
1 & & 1 & \\
1 & \end{array}
\]

the two sets of 3 diagonal real numbers correspond to 3-vectors.

Gunaydin and Gursey, in J. Math.Phys. 14 (1973) 1651-1667, said:

“… the Zorn’s vector matrices …[of]… scalars … and 3-vectors …[are a]… realization of the split octonion algebra …

… the split octonion algebra contains divisors of zero and hence is not a division algebra …

We can represent the action of SO(8) on the split octonion basis \([s]\) … \([s] = [ u u^* ] … u = [ u_1 , u_2 , u_3 , u_0 ] \) …\[and]… we will construct the LSO(8) matrices that are in local triality with each other … given an element \(T^L\) in LSO(8) action on the octonions there exist unique \(T^R\) and \(T^P\) in LSO(8) such that …  
\((T^L x) y + x (T^R y) = T^P (xy)\) for all \(x,y\) in \(O\)  
… Since the group SO(8) is the “Lie multiplication group” of octonions (i.e., that every action of SO(8) on \(O\) can be represented by octonion multiplication) …
Lie multiplication algebra of the octonions is defined as the Lie algebra with the elements: \( LMO = \text{Der}(O) (+) L_O \_0 (+) R_O \_0 \)
where \( L_O \_0 \) and \( R_O \_0 \) correspond to multiplication from the left and the right by traceless (or imaginary) octonion units … the derivation (Lie) algebra of octonions is isomorphic to the Lie algebra of G2 … the automorphism group of …[ the octonions]…

… one can reformulate the … principle of global triality … as follows:

Given \( d^1 \) in SO(8) and \( d^2, d^3 \) in SO(8)
\[ (d^1 x) (d^2 y) = (d^3 (xy)^*) \]
where \( ^* \) denotes octonion conjugation …
we have cyclic symmetry between \( d^1, d^2 \) and \( d^3 \) …
Since given \( d^1, d^2, \) and \( d^3 \) are determined uniquely up to a sign, the subgroup of SO(8) x SO(8) x SO(8) consisting of elements which are in triality will form a twofold covering group of SO(8), i.e., it will be isomorphic to Spin(8). …”.

Physically,
the Zorn matrices of the three Fr(3,O) Freudenthal 56v, 56s+, and 56s- algebras correspond to the \( 8+8+8 = 24 \) orange vertices

and they produce three split octonions
and three elements \( T^L, T^R, \) and \( T^P \) of LSO(8) related by triality, so that
taken together they produce a Spin(8)
whose D4 Lie algebra is a second D4 in the 168 of the E6 in E8 root vector decomposition \( 240 = 72 + 168. \)
Therefore, the physics of the E6 in E8 decomposition $240 = 72 + 168$ is equivalent to the physics of the E8 Physics model decomposition $240 = 112 + 128$. They both have effectively:

- Two (bright and orange/dark yellow vertices) 24-vertex D4 root vectors, whose $4+4 = 8$ Cartan subalgebra dimensions correspond to the 8 E8 Cartan subalgebra dimensions and to the 6 E6 plus 2 G2 Cartan subalgebra dimensions, with the G2 being the automorphism group of the split octonions formed by the Zorn matrices.

- 64 $= 8 \times 8$ components (blue vertices) in 8 dimensions of 8 Dirac gammas.

- 64 $= 8 \times 8$ components (red vertices) in 8 dimensions of 8 fundamental first-generation fermion particles.

- 64 $= 8 \times 8$ components (green vertices) in 8 dimensions of 8 fundamental first-generation fermion antiparticles.

The E6 in E8 decomposition $240 = 72 + 168$ gives insight into the physical interpretation of the two D4 (dark yellow and bright yellow vertices):
the 24 bright yellow D4 vertices share the 72 E6 vertices with \{ 1, e \} components of the fermions and Dirac gammas and those components (through the physical spacetime being the Shilov boundary related to complex geometry related to the complex number interpretation of \{ 1, e \}) naturally correspond to Gravity and the geometry of the mass-producing Higgs mechanism;

the 24 orange D4 vertices share the 168 vertices outside E6 with the \{ i, j, k, ie, je, ke \} components of the fermions and Dirac gammas,
and those components correspond in the octonion identification of fundamental fermions to the red, blue, green up quarks and red, blue, green down quarks and so to the color SU(3) of the Standard Model, so it is natural for the D4 of the 24 bright yellow vertices to correspond to the gauge groups of the Standard Model. Further, the 24 orange D4 vertices are in the $168 = 56v + 56s^+ + 56s^-$ which contain the three Zorn matrices of split octonions, and, as Gunaydin and Gursey said in J. Math.Phys. 14 (1973) 1651-1667, “… Under the SU(3) subgroup of split G2 leaving $u_0$ and $u_0^*$ invariant, the three split octonions $(u_1, u_2, u_3)$ transform like a unitary triplet (quarks) …”, so that it is natural for the D4 of the 24 orange vertices to correspond to the Standard Model including its color SU(3).
The relationship between E8 root vector decompositions and coordinate dimension

8-dim coordinates give a natural E8 Physics model $240 = 112 + 128$
9-dim coordinates give a natural E6 in E8 $240 = 72 + 168$
10-dim coordinates give a natural $E_8 = H_4 + H_4$ $240 = 120 + 120$

seems similar what happens for the 120 vertices of the 600-cell

4-dim coordinates give a natural $120 = 24 + 96$
5-dim coordinates give a natural $120 = 20 + 40 + 60 = 60 + 60$

Since $H_4$ is the symmetry group of the 600-cell and the 120 vertices of the 600-cell are half of the 240 E8 root vectors, such a similarity may give further insight into the structure of E8.

The 600-cell structures are described by H. S. M. Coxeter, in his paper "Regular and Semi-Regular Polytopes II" (Math. Zeit. 188 (1985) 555-591, reprinted in "Kaleidoscopes: Selected Writings of H. S. M. Coxeter" (Wiley 1995)) where he uses 4-dim Golden ratio $t$ coordinates (his x-coordinates) to describe the 120 vertices as the permutations of $t, t, t, t^{(-2)}$ with even number of minus signs - there are $4 \times 16 / 2 = 32$ of these

$(t^2, t^{(-1)}, t^{(-1)}, t^{(-1)})$ with even number of minus signs - there are $4 \times 16 / 2 = 32$ of these

$(\sqrt{5}, 1, 1, 1)$ with an odd number of minus signs - there are $4 \times 16 / 2 = 32$ of these

and

$(+/-2, +/-2, 0, 0)$ there are 24 of these

so that the natural decompositon is $120 = 32 + 32 + 32 + 24 = 96 + 24$
which corresponds to the 96 edges plus 24 vertices of a 24-cell.

Then to transform to 5-dim coordinates (his u-coordinates) he starts
with x-coordinates and adds fifth coordinate zero \((x_1, x_2, x_3, x_4, 0)\) and makes the transformation

\[
\begin{align*}
2u_1 &= -x_1 + t^2 x_2 + t^{(-2)} x_3 \\
2u_2 &= t^{(-2)} x_1 - x_2 + t^2 x_3 \\
2u_3 &= t^2 x_1 + t^{(-2)} x_2 - x_3 \\
2u_4 &= -x_1 - x_2 - x_3 + \sqrt{5} x_4 \\
2u_5 &= -x_1 - x_2 - x_3 - \sqrt{5} x_4
\end{align*}
\]

In terms of the \(u\)-coordinates, Coxeter gets a natural \(120 = 20 + 40 + 60 = 60 + 60\) decomposition, saying:

“… the 120 vertices … are the permutations of

\[
(\sqrt{5}, 0, 0, 0, -\sqrt{5}) \quad [\ 20 \ of \ these \ ]…
\]

\[
(t^2, t^{(-2)}, -1, -1, -1) \quad [\ 20 \ of \ these \ ]…
\]

\[
(1, 1, 1, -t^{(-2)}, -t^2) \quad [\ 20 \ of \ these \ ]…
\]

\[
(2, t^{(-1)}, t^{(-1)}, -t, -t) \quad [\ 30 \ of \ these \ ]…
\]

\[
(t, t, -t^{(-1)}, -t^{(-1)}, -2) \quad [\ 30 \ of \ these \ ]…”.
\]

More insight into these structures can be gained by considering what Conway and Sloane, in Chapter 8 of their book "Sphere Packings, Lattices and Groups" (Springer, 3rd ed 1999), said:

"... The icosian group is a multiplicative group of order 120 consisting of the quaternions ... where \((a, b, c, d)\) means \(a + bi + cj + dk\ ..."

\[
(1/2)( +/-2, 0, 0, 0 )
\]

\[
(1/2)( +/-1, +/-1, +/-1, +/-1 )
\]

\[
(1/2)( 0, +/-1, +/-s, +/-t )
\]

... all even permutations of the coordinates are permitted ...

\[
s = (1/2)(1 - \sqrt{5}) \quad , \quad t = (1/2)(1 + \sqrt{5})
\]
... The 240 minimal vectors of ... E8 ... consist of the elements q and s q where q is any element of the icosian group ...

[ The three sets of icosians have 8, 16, and 96 elements, and the first two sets form the 8+16 = 24 vertices of a 24-cell and the third set forms the 96 Golden Ratio points on the 96 edges of that 24-cell, corresponding to the 4-dim decomposition of the 120 vertices of the 600-cell as described above. ]

... We use the particular names
w = (1/2)( -1, 1, 1, 1 )
i_H = (1/2)( 0, 1, s, t )
There is a homomorphism from this [icosian] group to the Alternating group A5 on five letters \{G,H,I,J,K\} defined by
\[i = (1/2)( 0, 2, 0, 0 ) \rightarrow (H,I)(J,K)\]
\[j = (1/2)( 0, 0, 2, 0 ) \rightarrow (H,J)(K,I)\]
\[k = (1/2)( 0, 0, 0, 2 ) \rightarrow (H,K)(I,J)\]
w = (1/2)( -1, 1, 1, 1 ) \rightarrow (I,J,K)
i_H = (1/2)( 0, 1, s, t ) \rightarrow (G,I)(J,K)
in which the kernel is \{+/-1\}. Table 8.1 below gives much more information about this homomorphism. Abstractly the [120-element] icosian group is the perfect double cover 2.A5 of [60-element] A5, and is sometimes called the binary icosahedral group.
The icosian ring I is the set of all finite sums \(q_1 + \ldots + q_n\) where each \(q_i\) is in the icosian group. Elements of the icosian ring are ... called icosians.
The typical icosian \(q\) has the form \(q = a + bi + cj + dk\) where the coordinates \(a,b,c,d\) belong to the golden field \(Q(t)\) and so have the form \(x + y \sqrt{5}\) where \(x,y\) are in \(Q\) [the rational numbers].
The conjugate icosian ... is \(qbar = a - bi - cj - dk\) and \(q qbar = a^2 + b^2 + c^2 + d^2\)
Two vectors ... \(v = ( q_1, q_2, \ldots ) \) ... and \(w = ( r_1, r_2, \ldots ) \) have a quaternionic inner product
\[(v,w) = q_1 r_1bar + q_2 r_2bar + \ldots\]
We shall use two different norms for such vectors, the quaternionic norm
\[QN(v) = (v,v)\]
which is a number of the form \(a + b \sqrt{5}\) with \(a,b\) in \(Q\) and the Euclidean norm
\[EN(v) = a + b\]
The icosians of quaternionic norm 1 are the elements of the icosian group. With respect to the quaternionic norm the icosians belong to a four-dimensional space over $\mathbb{Q}(t)$; with the Euclidean norm they lie in an eight-dimensional space. In fact under the Euclidean norm the icosian ring $I$ is isomorphic to an $E_8$ lattice in this space. … Table 8.1 has 60 entries, one for each pair of elements $+/q$ of the icosian group. …
The typical entry in this table: \( \text{wbar}^i = w_{IG} \rightarrow (HJK) \) … should be read as follows. The top line gives name(s) for \( q \) (in this case \( \text{wbar}^i = w_{IG} = (1/2)( -1 - i + j + k ) \)) and indicates the corresponding even permutation of \{G,H,I,J,K\}.

The four quaternionic coordinates of \( 2q \) appear in the first column, followed by two columns giving the E8 vectors representing \( 2q \) and \( 2s \, q \)

As usual, - stands for -1 and + for +1.
The formulae …

\[ w^i = i^{-(1)} w \]
\[ w = w k = wbar + i = (1/2)( -1 + i - j - k ) \]
\[ w^j = j^{-(1)} w \]
\[ w = w k = wbar + j = (1/2)( -1 - i + j - k ) \]
\[ w^k = k^{-(1)} w \]
\[ w = w k = wbar + k = (1/2)( -1 - i - j + k ) \]
\[ wbar^i = i^{-(1)} wbar \]
\[ wbar = - k wbar = - wbar j = w - i = (1/2)( -1 - i + j + k ) \]
\[ wbar^j = j^{-(1)} wbar \]
\[ wbar = - i wbar = - wbar k = w - j = (1/2)( -1 + i - j + k ) \]
\[ wbar^k = k^{-(1)} wbar \]
\[ wbar = - j wbar = - wbar i = w - k = (1/2)( -1 + i + j - k ) \]

… are helpful for manipulating these quaternions.

The naming system for \( q = (1/2)( a, b, c, d ) \) is as follows.
The letters \( i,j,k \) indicate that \( a = 0 \)
\( w \) indicates \( a = -1 \) …[and]… \( s \) indicates \( a = -s \) …[and]… \( t \) indicates \( a = -t \)
and the subscripts indicate the signs of \( a,b,c,d \):

<table>
<thead>
<tr>
<th>signs</th>
<th>subscript</th>
</tr>
</thead>
<tbody>
<tr>
<td>0(+00)</td>
<td>G</td>
</tr>
<tr>
<td>-+++</td>
<td>GH</td>
</tr>
<tr>
<td>----</td>
<td>HG</td>
</tr>
<tr>
<td>-+-</td>
<td>GI</td>
</tr>
<tr>
<td>--++</td>
<td>IG</td>
</tr>
<tr>
<td>0+++</td>
<td>H</td>
</tr>
<tr>
<td>0+-</td>
<td>I</td>
</tr>
<tr>
<td>-0++</td>
<td>HI</td>
</tr>
<tr>
<td>-0--</td>
<td>IH</td>
</tr>
<tr>
<td>-0+-</td>
<td>JK</td>
</tr>
<tr>
<td>-0+</td>
<td>KJ</td>
</tr>
</tbody>
</table>

For example \( w_{XY} \) corresponds to the permutation \( (Z,T,U) \) where \( X,Y,Z,T,U \)
is an even permutation of \( G,H,I,J,K \). The triple \( \{ i_X, j_X, k_X \} \) where
\( X \) is any of \( G,H,I,J,K \) forms a system of unit quaternions
( with \( i_X^2 = -1, i_X j_X = k_X \), etc ).
The 240 minimal vectors of this version of E8 have Euclidean norm 1, and quater-
nionic norm either 1 or \( s^2 \). They consist of the elements \( q \) and \( s q \), where \( q \) is any
element of the icosian group. …". 
The 5-fold symmetry of the 120 Icosians and the 240 E8 Root Vectors that are formed by two sets of H4 Root Vectors (600-cell vertices), one set being the other dilated by the Golden Ratio, is seen in this Decagon Projection of Bathsheba Grossman’s E8 Glass model:

I added the red Pentagram that shows the Golden Ratio Dilated set of 120 vertices as being 10+30+10+60 outside the inner pentagon plus 10 at the center and the Undilated set of 120 vertices as being 10+30+10+60 inside the inner pentagon plus another 10 at the center.
More detail can be seen by looking at each set of 120 Icosian vertices by itself as the set of vertices of a 600-cell, and using these 600-cell images constructed by a java applet by Michael Gibbs:

a color stereo view

and a larger black-and-white mono view

The mathematical structure is related to a talk given in early 2008 at the University of California Riverside by Bertram Kostant. In his notes of the talk, John Baez said: “…The Cartan subalgebra … h … for the Lie algebra e8 … is 8-dimensional, and there are 240 roots, so the dimension of e8 is 248 … the 248-dimensional Lie
algebra \( e_8 \) is a direct sum of 31 8-dimensional Cartan algebras. …

There's … a copy of \((\mathbb{Z}/5)^3\) in \( E_8 \). If we think of this as a 3-dimensional vector space containing "lines" (1-dimensional subspaces), then it contains

\[
1 + 5 + 5^2 = 31
\]

lines. The centralizer in \( E_8 \) of any such line is

\[
\text{SU}(5) \times \text{SU}(5)
\]

This group is 48-dimensional. It has a 248-dimensional representation coming from the adjoint action on \( e_8 \). This is the direct sum of the 48-dimensional subrepresentation coming from \( \text{su}(5) \) in \( e_8 \) and a representation of dimension \( 248 - 48 = 200 \) …”.

From the point of view of my \( E_8 \) Physics model, the 200 breaks down into two copies of 100, and each 100 breaks down into 12 + 24+32+32, and the 40 Root Vector Vertices assigned by John Baez to two copies of \( \text{SU}(5) \) are seen as the 40 Root Vector Vertices of 45-dim \( D_5 \):

\[
\begin{align*}
8+8+8 & = 24 \text{ for a 4-dim 24-cell ( the Root Vectors of a } D_4 \text{ in } D_5 \text{ ) } \\
8 & = 8 \text{ for a 4-dim HyperOctahedron above (in a 5 th dim) the 24-cell } \\
8 & = 8 \text{ for a 4-dim HyperOctahedron below (in 5 th dim) the 24-cell } \\
8+8+8+8 & = 5\times8 = 40 \text{ Root Vector Vertices for } D_5.
\end{align*}
\]

\( D_5 \) is a Lie subalgebra of the \( E_6 \) Lie subalgebra of the \( E_8 \) Lie algebra, with 24 of the 40 representing the Root Vectors of a \( D_4 \) containing a \( D_3 \) Conformal Group Lie subalgebra for MacDowell-Mansouri Gravity and the 40-24 = 16 representing a Complex 8-dim Kaluza-Klein Spacetime whose 8-Real-dim Shilov Boundary is a Kaluza-Klein Physical Spacetime plus Internal Symmetry Space.

The 24-12 = 12 Root Vectors of \( D_4 \) not in the \( D_3 \) Conformal Group represent the 12 generators of \( S(U(3)\times U(2)) \) as to which John Baez said in his notes on Bertram Kostant’s U. C. Riverside 2008 lecture:

“…The gauge group of the Standard model is usually said to be \( \text{SU}(3)\times\text{SU}(2)\times\text{U}(1) \), but this group has a \( \mathbb{Z}/6 \) subgroup that acts trivially on all known particles. The quotient \( (\text{SU}(3)\times\text{SU}(2)\times\text{U}(1))/(\mathbb{Z}/6) \) is isomorphic to
S(U(3)xU(2)) - that is, the subgroup of SU(5) consisting of block diagonal matrices with a 3x3 block and a 2x2 block.
So … S(U(3)xU(2)) …
could be called the "true" gauge group of the Standard Model. …”.

Color-code the Decagon Projection vertices by:
Orange/Yellow for 24+24 = 48 for the two copies of D4 in D8 in E8,
each D4 giving Gauge Bosons for Gravity and the Standard Model,
and the Yellow D4 being inside D5 which has 24+16 = 40 Root Vectors;
Blue for 16 for Complex-8-dim-Spacetime of D5;
Blue for 6x8 = 48 for 8-dim Spacetime components with respect to the 3 colors and 3 anticolors of the SU(3) Color Force, which, combined with the two Complex components of the 16 Blue inside D5, give all 8 Octonionic components of 8-dim Spacetime;
Red for 8x8 = 64 for the 8 Octonionic components of the 8 First-Generation Fundamental Fermion Particles;
Green for 8x8 = 64 for the 8 Octonionic components of the 8 First-Generation Fundamental Fermion AntiParticles.

Then, the 48 + 200 breakdown 248-dim E8 described above by John Baez gives a 40 + 200 breakdown of the 240 Root Vector Vertices of E8, which in turn can be broken down into
an inner set of 20 + 100 = 120 = 10+(2+8) + 12 + 24+32+32
(the (2+8) being a Central 10 at the zero-radius center of the Decagon Projection, the 10 and 2 and 12 being the 24 of a D4, the 8 and 24 being 32 of 64 Octonionic components of 8 Spacetime Dimensions, a 32 being 32 of 64 Octonionic components of 8 Fermion Particles, the other 32 being 32 of 64 Octonionic components of 8 Fermion Particles)
and
an outer set of 20 + 100 = 120 = 10+(2+8) + 12 + 24+32+32
(the (2+8) being a Central 10 at the zero-radius center of the Decagon Projection, the 10 and 2 and 12 being the 24 of a D4, the 8 and 24 being 32 of 64 Octonionic components of 8 Spacetime Dimensions, a 32 being 32 of 64 Octonionic components of 8 Fermion Particles, the other 32 being 32 of 64 Octonionic components of 8 Fermion Particles)
with
the Outer 120 vertices being the Inner 120 dilated by the Golden Ratio, and the mapping between the Inner 120 and the Outer 120 corresponding to the {-1,+1} of the Electroweak U(2) Charges.
The Inner 120 and Outer 120 look like