F4 and E8: 
Commutators and AntiCommutators

Frank Dodd (Tony) Smith Jr. - 2013

Abstract:

Realistic Physics models must describe both commutator Bosons and anticommutator Fermions so that spin and statistics are consistent. The usual commutator structure of Lie Algebras can only describe Bosons, so a common objection to Physics models that describe both Bosons and Fermions in terms of a single unifying Lie Algebra (for example, Garrett Lisi's E8 TOE) is that they violate consistency of spin and statistics by using Lie Algebra commutators to describe Fermions. However, Pierre Ramond has shown in hep-th/0112261 as shown that the exceptional Lie Algebra F4 can be described using anticommutators as well as commutators. This essay uses the periodicity property of Real Clifford Algebras to show that E8 can also be described using anticommutators as well as commutators so that it may be possible to construct a realistic Physics model that uses the exceptional Lie Algebra E8 to describe both Bosons and Fermions.

E8 also inherits from F4 Triality-based symmetries between Bosons and Fermions that can give the useful results of SuperSymmetry without requiring conventional SuperPartner particles that are unobserved by LHC.
Realistic Physics models must describe both integer-spin Bosons whose statistics are described by commutators (examples are Photons, W and Z bosons, Gluons, Gravitons, Higgs bosons) and half-integer-spin Fermions whose statistics are described by anticommutators. (examples are 3 generations of Electrons, Neutrinos, Quarks and their antiparticles)

Lie Algebra elements are usually described by commutators of their elements so if a Physics model attempts to describe both Bosons and Fermions as elements of a single unifying Lie Algebra (for example, Garrett Lisi’s E8 TOE) a common objection is:

since the Lie Algebra is described by commutators, it can only describe Bosons and cannot describe Fermions therefore models (such as Garrett Lisi’s) using E8 as a single unifying Lie Algebra violate the consistency of spin and statistics and are wrong.

However, Pierre Ramond has shown in hep-th/0112261 as shown that the exceptional Lie Algebra F4 can be described using anticommutators as well as commutators.

The periodicity property of Real Clifford Algebras shows that E8 inherits from F4 a description using anticommutators as well as commutators so that it may be possible to construct a realistic Physics model that uses the exceptional Lie Algebra E8 to describe both Bosons and Fermions.

Here are relevant quotes from hep-th/0112261 by Pierre Ramond: "... exceptional algebras relate tensor and spinor representations of their orthogonal subgroups, while Spin-Statistics requires them to be treated differently ... all representations of the exceptional group F4 are generated by three sets of oscillators transforming as 26. We label each copy of 26 oscillators as $A_{0}^{[k]}, A_{i}^{[k]}, i = 1, \ldots, 9, B_{a}^{[k]}, a = 1, \ldots, 16,$ and their hermitian conjugates, and where $k = 1, 2, 3$. Under SO(9), the $A[k]_{i}$ transform as 9, $B[k]_{a}$ transform as 16, and $A[k]_{0}$ is a scalar. They satisfy the commutation relations of ordinary harmonic oscillators
Note that the SO(9) spinor operators satisfy Bose-like commutation relations

\[ [A^{[\kappa]}_i, A^{[\kappa']\dagger}_j] = \delta_{ij} \delta^{[\kappa][\kappa']} \quad \text{and} \quad [A^{[\kappa]}_0, A^{[\kappa']\dagger}_0] = \delta^{[\kappa][\kappa']} \]

The generators \( T_{ij} \) and \( T_a \)

\[
T_{ij} = -i \sum_{\kappa=1}^{4} \left\{ (A^{[\kappa]\dagger}_i A^{[\kappa]}_j - A^{[\kappa] \dagger}_j A^{[\kappa]}_i) + \frac{1}{2} B^{[\kappa]\dagger} \gamma_{ij} B^{[\kappa]} \right\} ,
\]

\[
T_a = -\frac{i}{2} \sum_{\kappa=1}^{4} \left\{ (\gamma_i)^{ab} \left( A^{[\kappa]\dagger}_i B^{[\kappa]}_b - B^{[\kappa]\dagger}_b A^{[\kappa]}_i \right) - \sqrt{3} \left( B^{[\kappa]\dagger}_a A^{[\kappa]}_0 - A^{[\kappa]\dagger}_0 B^{[\kappa]}_a \right) \right\} ,
\]

satisfy the F4 algebra,

\[
[T_{ij}, T_{kl}] = -i \left( \delta_{jk} T_{il} + \delta_{il} T_{jk} - \delta_{ik} T_{jl} - \delta_{jl} T_{ik} \right) ,
\]

\[ [T_{ij}, T_a] = \frac{i}{2} (\gamma_{ij})_{ab} T_b , \]

\[ [T_a, T_b] = \frac{i}{2} (\gamma_{ij})_{ab} T_{ij} , \]

so that the structure constants are given by

\[ f_{ijab} = f_{abij} = \frac{1}{2} (\gamma_{ij})_{ab} . \]

The last commutator requires the Fierz-derived identity

\[ f_{ijab} = f_{abij} = \frac{1}{2} (\gamma_{ij})_{ab} . \]

from which we deduce

\[ 3 \delta^{ac} \delta^{db} + (\gamma^{i})^{ac} (\gamma^{i})^{db} - (a \leftrightarrow b) = \frac{1}{4} (\gamma^{ij})^{ab} (\gamma^{ij})^{cd} . \]

To satisfy these commutation relations, we have required both \( A_0 \) and \( B_a \) to obey Bose commutation relations

(\text{Curiously, if both } [A_0 \text{ and } B_a \text{ are anticommuting, the F4 algebra is still satisfied}). \ldots"
The $1 + 9 + 16 = 26$ oscillators
\[ A_0^{[\kappa]}, \ A_i^{[\kappa]}, \ i = 1, \cdots, 9, \ B_a^{[\kappa]}, \ a = 1, \cdots, 16, \]
represent the 26-dim lowest-dimensional non-trivial representation of 52-dim $F_4$.

The $36 + 16 = 52$ generators
\[
T_{ij} = -i \sum_{\kappa=1}^{4} \left\{ (A_i^{[\kappa]\dagger} A_j^{[\kappa]} - A_j^{[\kappa]\dagger} A_i^{[\kappa]}) + \frac{1}{2} B^{[\kappa]\dagger} \gamma_{ij} B^{[\kappa]} \right\},
\]
\[
T_a = -\frac{i}{2} \sum_{\kappa=1}^{4} \left\{ (\gamma_i)^{ab} \left( A_i^{[\kappa]\dagger} B_b^{[\kappa]} - B_b^{[\kappa]\dagger} A_i^{[\kappa]} \right) - \sqrt{3} \left( B_a^{[\kappa]\dagger} A_0^{[\kappa]} - A_0^{[\kappa]\dagger} B_a^{[\kappa]} \right) \right\},
\]
represent the 52-dim adjoint representation of $F_4$ written as commutators.

By the remark shown in bold in the quote above, Ramond states that the 16 Spinor oscillators $B[k]_a$ can be written as anticommutators as well as commutators and that both cases produce the 52-dim $F_4$ algebra.

Physically, this means that if you use the 52-dim $F_4$ to build a physics model with Fermions being represented by $16$-dim $F_4 / SO(9) = O(2)$ then you can use the anticommutator structure of the 16-dim $B[k]_a$ to satisfy the spin-statistics theorem because the $B[k]_a$ represent a 16 of $SO(9)$ which is also $O(2) = F_4 / SO(9)$.

To see how this anticommutator structure extends to $E_8$, note that the lowest-dimensional non-trivial representation of $E_8$ is 248-dim and that the adjoint representation of $E_8$ is also 248-dim.
(quotation slightly modified due to typographical considerations etc):

"... Schwinger ... has studied the generators and irreps of SU(2) in terms of two
Bose oscillators (hereafter abbreviated SHO) ... the algebras of the classical
groups ... A(n) ; B(n) ; D(n) ; C(n) have been realized in terms of Fermi oscillators.
The spinor irreps of the orthogonal groups ... are the only ones which ... have been
constructed ... using Fermi oscillators. These irreps involve various numbers of
Fermi oscillator creation operators a†_i

... the elementary spinor irreps of the orthogonal groups can be written in terms of a
single SHO creation operator acting on the vacuum state ... the 'vacuum state' does
not have zero weight, but is an element of a spinor irrep.

... For D(n), the 'vacuum state', together with all states formed by even powers of a†
operating on this state, up to the maximum possible such power, constitute the
elements of one of the elementary spinor irreps; all possible odd powers of a†,
operating on the 'vacuum', constitute the set of all elements of the other elementary
spinor irrep ...

... F(4) ... we have elementary irrep f1 = 26-dim
so that D(13) = SO(26) contains F4
The other elementary irrep of F(4) is f2 = 52-dim

... E(8) ... we have elementary irrep f1 = 248-dim
The simplest embedding of E(8) is to choose D(124) = SO(248) contains E(8) ...

Physically, if you use the 248-dim E8 to build a physics model
with Fermions being represented by 128-dim E8 / D8 = (OxO)P2
and
if E8 inherits anticommutator structure from F4
then
you can use anticommutator structure of the 128-dim (OxO)P2
to satisfy the spin-statistics theorem.
To see how the anticommuting property of the 16 $B_a$ elements of $F_4$ can be inherited by some of the elements of $E_8$, consider that 52-dimensional $F_4$ is made up of:

- 28-dimensional $D_4$ Lie Algebra Spin(8) (in commutator part of $F_4$)
- 8-dimensional $D_4$ Vector Representation $V_8$ (in commutator part of $F_4$)
- 8-dimensional $D_4$ +half-Spinor Representation $S^+_8$ (in anticommutator part of $F_4$)
- 8-dimensional $D_4$ -half-Spinor Representation $S^-_8$ (in anticommutator part of $F_4$)

Since 28-dimensional $D_4$ Spin(8) is the BiVector part $BV_{28}$ of the Real Clifford Algebra $Cl(8)$ with graded structure:

$$Cl(8) = 1 + V_8 + BV_{28} + 56 + 70 + 56 + 28 + 8 + 1$$

and with Spinor structure:

$$Cl(8) = (S^+_8 + S^-_8) \times (8 + 8)$$

$F_4$ can be embedded in $Cl(8)$ (blue commutator part, red anticommutator part):

$$F_4 = V_8 + BV_{28} + S^+_8 + S^-_8$$

Note that $V_8$ and $S^+_8$ and $S^-_8$ are related by the Triality Automorphism.
Also consider the 8-periodicity of Real Clifford Algebras, according to which for all N

\[
\text{Cl}(8N) = \text{Cl}(8) \times \ldots (N \text{ times tensor product}) \ldots \text{Cl}(8)
\]

so that in particular \( \text{Cl}(16) = \text{Cl}(8) \times \text{Cl}(8) \)

where \( \text{Cl}(16) \) graded structure is \( 1 + 16 + \text{BV}_{120} + 560 + \ldots + 16 + 1 \)

and \( \text{Cl}(16) \) Spinor structure is \( \left( (\text{S}_{64} + \text{S}_{-64}) + (64 + 64) \right) \times (128 + 128) \)

and \( \text{Cl}(16) \) contains 248-dimensional \( \text{E}_8 \) as

\[
\text{E}_8 = \text{BV}_{120} + \text{S}_{+64} + \text{S}_{-64}
\]

where \( \text{BV}_{120} = 120\)-dimensional \( \text{D}_8 \) Lie Algebra \( \text{Spin}(16) \)

and \( \text{S}_{+64} + \text{S}_{-64} = 128\)-dimensional \( \text{D}_8 \) half-Spinor Representation

Consider two copies of \( \text{F}_4 \) embedded into two copies of \( \text{Cl}(8) \).

**For commutator structure:**

The tensor product of the two copies of \( \text{Cl}(8) \) can be seen as

\[
1 + \text{V}_8 + \text{BV}_{28} + 56 + 70 + 56 + 28 + 8 + 1
\]

\[
\times
\]

\[
1 + \text{V}_8 + \text{BV}_{28} + 56 + 70 + 56 + 28 + 8 + 1
\]

which produces the Real Clifford Algebra \( \text{Cl}(16) \) with graded structure

\[
1 + 16 + \text{BV}_{120} + 560 + 1820 + \ldots + 16 + 1
\]

where the \( \text{Cl}(16) \) BiVector \( \text{BV}_{120} \) is made up of 3 parts

\[
\text{BV}_{120} = \text{BV}_{28} \times 1 + 1 \times \text{BV}_{28} + \text{V}_8 \times \text{V}_8
\]

that come from the \( \text{V}_8 \) and \( \text{BV}_{28} \) commutator parts of the two copies of \( \text{F}_4 \).

This gives the commutator part of \( \text{E}_8 \) as \( \text{BV}_{120} \) inheriting commutator structure from the two copies of \( \text{F}_4 \) embedded in two copies of \( \text{Cl}(8) \) whose tensor product produces \( \text{Cl}(16) \) containing \( \text{E}_8 \).
**For anticommutator structure:**

The tensor product of the two copies of 256-dim Cl(8) can also be seen as

\[
( (S^8 + S^{-8}) \times (8 + 8) )
\]

\[
\times
\]

\[
( (S^8 + S^{-8}) \times (8 + 8) )
\]

which produces the \(2^{16} = 65,536 = 256 \times 256\)-dim Real Clifford Algebra Cl(16)

\[
( (S^8 + S^{-8}) \times (S^8 + S^{-8}))
\]

\[
\times
\]

\[
( (8 + 8) \times (8 + 8) )
\]

with 256-dimensional Spinor structure

\[
( (S^8 + S^{-8}) \times (S^8 + S^{-8})) =
\]

\[
= ( (S^8 \times S^8) + (S^{-8} \times S^{-8}) ) + ( (S^8 \times S^{-8}) + (S^{-8} \times S^8) )
\]

that comes from the \(S^8\) and \(S^{-8}\) anticommutator parts of the two copies of F4.

Since the \((S^8 \times S^{-8})\) and \((S^{-8} \times S^8)\) terms inherit mixed helicities from F4

only the \((S^8 \times S^8)\) and \((S^{-8} \times S^{-8})\) terms inherit consistent helicity from F4.

Therefore, define \(S^+64 = (S^8 \times S^8)\) and \(S^-64 = (S^{-8} \times S^{-8})\)

so that

\[
( S^+64 + S^-64 ) = 128\text{-dimensional D8 half-Spinor Representation}
\]

This gives the anticommutator part of E8 as \(S^+64 + S^-64\) inheriting anticommutator structure from the two copies of F4 embedded in two copies of Cl(8) whose tensor product produces Cl(16) containing E8.
The result is that 248-dimensional E8 is made up of:

- $\text{BV120} = 120$-dimensional D8 Lie Algebra Spin(16) (commutator part of E8)
- 128-dimensional $(\text{S}+64 + \text{S}-64)$ D8 half-Spinor (anticommutator part of E8)

Note that since the $\text{V8}$ and $\text{S+8}$ and $\text{S-8}$ components of F4 are related by Triality, and since
the E8 component $\text{BV120}$ contains 64-dimensional $\text{V8} \times \text{V8}$
and
the 64-dimensional E8 component $\text{S+64} = \text{S+8} \times \text{S+8}$
and
the 64-dimensional E8 component $\text{S-64} = \text{S-8} \times \text{S-8}$

E8 inherits from the two copies of F4 a Triality relation

$$\text{V8} \times \text{V8} = \text{S+64} = \text{S-64}$$

The commutator - anticommutator structure of E8 allows construction of realistic Physics models that not only unify both Bosons and Fermions within E8 but also contain Triality-based symmetries between Bosons and Fermions that can give the useful results of SuperSymmetry without requiring conventional SuperPartner particles that are unobserved by LHC.

**CONCLUSION:**

Unified E8 Physics models can be constructed without violating spin-statistics, so that evaluation of such models as Garrett Lisi's E8 TOE and my E8 Physics model at http://vixra.org/abs/1108.0027 should be based on other criteria such as consistency with experimental observations.
Cl(16) contains E8

\[ Cl(16) = 1 + 16 + 120 + 560 + 1820 + ... + 16 + 1 = \]
\[ = (64++ + 64--) + (64+- + 64-) \times 256 \]

Cl(16) = Cl(8)₁ x Cl(8)₂ where Cl(8)₁ contains F4₁ and Cl(8)₂ contains F4₂

\[ (8\text{of}F4₁ \times 8\text{of}F4₂) + 28\text{of}F4₁\text{x1of}Cl(8)₂ + 28\text{of}F4₂\text{x1of}Cl(8)₁ = 64 + 28 + 28 = 120\text{of}E8 \]

\[ (8+\text{of}F4₁ + 8-\text{of}F4₁) \times (8+\text{of}F4₂ + 8-\text{of}F4₂) \text{ gives } (64++ + 64--)\text{of}E8 \]

(because the terms (64++ + 64--) are rejected as unphysical mixed helicity)

( The notation -> denotes a Maximal Contraction )

E8 = 120 + (64++ + 64--)

E8 -> A7 = SU(8) part of U(8) for 8 position x 8 momentum = 64 generators -

semidirect product with -

28 + 64 + 1 + 64 + 28 Heisenberg Algebra H92 for

28 gauge bosons for A2xA1xA0 = SU(3)xSU(2)xU(1) plus

D3 = Spin(2,4) = A3 = SU(2,2) Conformal Gravity

and for

64 = 8x8 of 8 components of 8 Fermions

E7.5 ->E7 - semidirect product with - 28+1+28 Heisenberg Algebra H28

28 = Quaternionic Jordan Algebra J4(Q) =bijection= D4

D4 = Spin(8) = (Standard Model + Gravity) Gauge Bosons

E7 -> E6 - semidirect product with - 27+1+27 Heisenberg Algebra H27

27 has 3 diagonal generators of Quaternion SU(2) + (8 + 8) Fermions) + 8 Spacetime vectors

27 = Octonionic Jordan Algebra J3(O) =bijection= J4(Q)o

E6 -> D5 = Spin(10) - semidirect product with - 16+1+16 Heisenberg Algebra H16

16 = (26-10)-dim Fermion part of J3(O)o

E6 = F4 + J3(O)o 26-dim traceless part of 27-dim Jordan Algebra J3(O)

F4 = 8 + 28 + (8+ + 8-)

F4 -> B3 = (Spin(6) + S6) - semidirect product with - 15+1+15 Heisenberg Algebra H15

for 15 D3 = Spin(2,4) Conformal Gauge Bosons

6-dim Conformal Spacetime reduces to M4 Physical Spacetime

Cl(8) = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = (8+ + 8-) x 16