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HARMONIC ANALYSIS

of Functions of Several Complex Variables in the

CLASSICAL DOMAINS

by

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ГАРМОНИЧЕСКИЙ АНАЛИЗ
ФУНКЦИЙ МНОГИХ КОМПЛЕКСНЫХ ПЕРЕМЕННЫХ
В КЛАССИЧЕСКИХ ОБЛАСТЯХ

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INTRODUCTION

I. Classical domains. By a classical domain we shall understand an irreducible bounded symmetric domain (in the space of several complex variables) of one of the following four types:

(1) The domain $\mathfrak{R}_{\mathrm{I}}$ of $m \times n$ matrices with complex entries satisfying the condition

$$I^{(m)} - Z\overline{Z'} > 0.$$

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Here $I^{(m)}$ is the identity matrix of order m, $\overline{Z'}$ is the complex conjugate of the transposed matrix Z'. ($H > 0$ for a hermitian matrix H denotes, as usual, that H is positive definite.)

(2) The domain $\mathfrak{R}_{\mathrm{II}}$ of symmetric matrices of order n (with complex entries) satisfying the condition

$$I^{(n)} - Z\overline{Z} > 0.$$

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(3) The domain $\mathfrak{R}_{\mathrm{III}}$ of skew-symmetric matrices of order n (with complex entries) satisfying the condition

$$I^{(n)} + Z\overline{Z} > 0.$$

DIII

(4) The domain $\mathfrak{R}_{\mathrm{IV}}$ of n-dimensional ($n > 2$) vectors

$$z = (z_1, z_2, \ldots, z_n)$$

BDI (q=2)

(z_k are complex numbers) satisfying the conditions[2]

$$|zz'|^2 + 1 - 2z\overline{z}' > 0, \qquad |zz'| < 1.$$

The complex dimension of these four domains is mn, $n(n+1)/2$, $n(n-1)/2$, n, respectively.

The author has shown (cf. L. K. Hua [3]) that $\mathfrak{R}_{\mathrm{IV}}$ can also be regarded as a homogeneous space of $2 \times n$ real matrices. Therefore, the study of all these domains can be reduced to a study of the geometry of matrices.

In 1935, E. Cartan [1] proved that there exist only six types of irreducible homogeneous bounded symmetric domains. Beside the above four types, there exist only two; their dimensions are 16 and 27. Of course

[2] Translator's note (n.b., unless otherwise noted, these words refer to the Russian translator). Here and throughout, the author considers a vector as a matrix of one row and n columns. So z' is a matrix of one column and n rows (the transpose of the matrix z).

these two types are rather special. The problem of the explicit description of these two types is still open.

The purpose of the present book is to study harmonic analysis on the classical domains. (The exact content of this harmonic analysis will be outlined later.)

II. Characteristic manifolds. Let $\mathfrak{R}$ be a bounded homogeneous domain in the $2n$-dimensional Euclidean space of n complex variables $z=(z_1, z_2, \cdots, z_n)$, and $f(z)$ an analytic function of z, regular in $\mathfrak{R}$. It is known that the maximum of the modulus of the function $f(z)$ is assumed on the boundary of $\mathfrak{R}$. Let $\mathfrak{C}$ be a manifold on the boundary of $\mathfrak{R}$ having the following properties: → Silov boundary

(a) The modulus of every analytic function regular in $\mathfrak{R}$ assumes its maximum on $\mathfrak{C}$.

(b) For every point a on $\mathfrak{C}$ there exists a function $f(z)$, regular on $\mathfrak{R}$, such that the modulus of $f(z)$ assumes its maximum at $z=a$.

Such a manifold $\mathfrak{C}$ is called a characteristic manifold of the domain $\mathfrak{R}$. We should mention that $\mathfrak{C}$ is in general a proper subset of the boundary, and that the dimension of $\mathfrak{C}$ may be much less than $2n-1$. It is clear that $\mathfrak{C}$ is uniquely determined by $\mathfrak{R}$. It is easy to show that $\mathfrak{C}$ is closed, and that an analytic function which is regular in a neighborhood of each point of $\mathfrak{C}$ is uniquely determined by its values on $\mathfrak{C}$. Hence it follows that the real dimension of $\mathfrak{C}$ is not less than n. We shall denote by ξ the variable on $\mathfrak{C}$, and by $d\xi \overline{d\xi}'$ and $\dot{\xi}$ the metric and the element of volume of $\mathfrak{C}$.

Clearly, in the definition of $\mathfrak{C}$ it is enough to consider only linear functions instead of all analytic functions.

We describe the characteristic manifolds of the classical domains.

(1) $\mathfrak{C}_{\mathrm{I}}$ consists of the $m \times n$ matrices U satisfying the condition

$$U\overline{U}' = I^{(m)}.$$

(2) $\mathfrak{C}_{\mathrm{II}}$ consists of all symmetric unitary matrices of order n.

(3) $\mathfrak{C}_{\mathrm{III}}$ is defined differently for even and odd n. If n is even, then $\mathfrak{C}_{\mathrm{III}}$ consists of all skew-symmetric unitary matrices of order n. If n is odd, then $\mathfrak{C}_{\mathrm{III}}$ consists of all matrices of the form

$$UDU',$$

where U is an arbitrary unitary matrix and

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus 0.$$

(4) $\mathfrak{C}_{\mathrm{IV}}$ consists of the vectors $e^{i\theta}x$, where x is a real vector such that $xx'=1$.

The manifolds $\mathfrak{C}_{\mathrm{I}}$, $\mathfrak{C}_{\mathrm{II}}$, $\mathfrak{C}_{\mathrm{III}}$ and $\mathfrak{C}_{\mathrm{IV}}$ have real dimension $m(2n-m)$, $n(n+1)/2$, $n(n-1)/2+(1+(-1)^n)(n-1)/2$ and n, respectively.

These characteristic manifolds are homogeneous spaces. Furthermore, any point of $\mathfrak{C}$ can be carried into any other point of $\mathfrak{C}$ by a transformation leaving a given point of $\mathfrak{R}$ invariant. The general theory of harmonic analysis on homogeneous spaces has been developed earlier (cf. E. Cartan [1], H. Weyl [1]); however, the method presented in this book gives more precise and more useful results.

2.5. The volume of $\mathfrak{R}_{IV}$. The domain $\mathfrak{R}_{IV}$ (the Lie-sphere) consists of all n-dimensional complex vectors z satisfying the conditions

$$|zz'|^2 + 1 - 2\bar{z}z' > 0, \tag{2.5.1}$$

$$|zz'| < 1. \tag{2.5.2}$$

We show that these two inequalities can be replaced by one. First we rewrite the inequalities (2.5.1) and (2.5.2) as

$$(1 - \bar{z}z')^2 > (\bar{z}z')^2 - |zz'|^2 > (\bar{z}z')^2 - 1. \tag{2.5.3}$$

From (2.5.3) it follows easily that

$$\bar{z}z' < 1. \tag{2.5.4}$$

The first of the inequalities (2.5.3) implies (since $\bar{z}z' \geq |zz'|$) that

$$1 - \bar{z}z' > \sqrt{(\bar{z}z')^2 - |zz'|^2}. \tag{2.5.5}$$

Therefore all vectors z satisfying the inequalities (2.5.1) and (2.5.2) also satisfy the inequality (2.5.5).

On the other hand, every vector satisfying (2.5.5) obviously satisfies (2.5.1). Furthermore from $|zz'| \leq \bar{z}z'$ and (2.5.5) we can deduce $|zz'| < 1$, i.e., (2.5.2). Therefore the domain $\mathfrak{R}_{IV}$ can be defined by the single inequality (2.5.5).

THEOREM 2.5.1. *For $\alpha > -1$ and $\beta > -(n+\alpha)$*

$$L_n(\alpha, \beta) = \int_{\mathfrak{R}_{IV}} \left(1 - \bar{z}z' - \sqrt{(\bar{z}z')^2 - |zz'|^2}\right)^{\alpha} \times \left(1 - \bar{z}z' + \sqrt{(\bar{z}z')^2 - |zz'|^2}\right)^{\beta} \dot{z} = \frac{\pi^n}{2^{n-1}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n)} \cdot \frac{1}{\alpha+\beta+n}. \tag{2.5.6}$$

In particular, for $\alpha = \beta = 0$ we obtain a formula for the volume of the domain $\mathfrak{R}_{IV}$,

$$V(\mathfrak{R}_{IV}) = \frac{\pi^n}{2^{n-1}} \cdot \frac{1}{n!}. \tag{2.5.7}$$

which holds for $\varphi(z)$ analytic in a closed circular domain with its center at the origin and z lying inside the domain.)

4°. The characteristic manifold of the domain $\mathfrak{R}_{IV}$ consists of vectors of the form $e^{i\theta}x$, where $0 \leq \theta \leq \pi$, and $x=(x_1, \cdots, x_n)$ is a real vector which satisfies the condition $xx'=1$.

$$H(z, \theta, x) = \frac{1}{V(\mathfrak{C}_{IV})\,[(x-e^{-i\theta}z)(x-e^{-i\theta}z)']^{n/2}}, \qquad (4.7.11)$$

It is easy to calculate the magnitude of the volume $V(\mathfrak{C}_{IV})$:

$$V(\mathfrak{C}_{IV}) = \frac{2\pi^{\frac{n}{2}+1}}{\Gamma\left(\frac{n}{2}\right)}.$$

4.8. The Poisson kernel for circular domains. Suppose that $\mathfrak{R}$, just as in §4.5, is a star-shaped circular domain, and $\mathfrak{C}$ its characteristic manifold, transitive with respect to the group Γ_0 of motions of $\mathfrak{R}$ which leave the origin unchanged. Then, by Theorem 4.6.1, there exists a Cauchy kernel

for the domain $\mathfrak{R}$, and Cauchy's formula holds for any function $f(z)$ which is analytic in $\mathfrak{R}$ and on its boundary.

Setting, in particular,

$$f(z) = H(z, \overline{w})\, g(z),$$

where $g(z)$ is an arbitrary function which is analytic in $\mathfrak{R}$ and on its boundary, we have

$$H(z, \overline{w})\, g(z) = \int_{\mathfrak{C}} H(z, \xi)\, H(\xi, \overline{w})\, g(\xi)\, \dot{\xi}.$$

For $w=z$, we obtain Poisson's formula

$$g(z) = \int_{\mathfrak{C}} P(z, \xi)\, g(\xi)\, \dot{\xi}, \qquad (4.8.1)$$

where the kernel

$$P(z, \xi) = \frac{H(z, \overline{\xi})\; H(\xi, \overline{z})}{H(z, \overline{z})} \qquad (4.8.2)$$

has occured above under the name of the Poisson kernel of the domain $\mathfrak{R}$.

Up to now we have established that formula (4.8.1) is valid for analytic $g(z)$; yet it can be extended to other classes of functions too (see §5.8).

closure of the linear span of the system $\{\varphi_\nu(z)\}$ (see §5.10).

If $\mathfrak{R}$ satisfies the conditions of Theorem 4.6.3, then the Poisson kernel can be written in the following simple form:

$$P(z, \xi) = \frac{1}{V(\mathfrak{C})} \cdot |\, B(\xi, z, U)\,|. \qquad (4.8.4)$$

In conclusion let us list the Poisson kernels for the classical domains.

Note that: Gamma(N+1) = N! for integer N; Gamma(1/2) = sqrt(pi); Gamma(N+1) = N Gamma(N)

In both cases $K \in \mathfrak{C}_{\mathrm{III}}$.

(4) For $\mathfrak{R}_{\mathrm{IV}}$

$$P(z, \xi) = \frac{1}{V(\mathfrak{C}_{\mathrm{IV}})} \cdot \frac{(1 + |zz'|^2 - 2\bar{z}z')^{\frac{n}{2}}}{|(z-\xi)(z-\xi)'|^n}, \tag{4.8.9}$$

where $\xi \in \mathfrak{C}_{\mathrm{IV}}$.