On the invariant sets of a family of quadratic maps

by

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Abstract

The Julia sets $B_\lambda$ belonging to a family of mappings $T : \mathbb{C} \rightarrow \mathbb{C}$ where $\lambda$ is a parameter is investigated. The mappings are defined by $T_\lambda z = (\lambda - z)^2$ for each $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$. For $\lambda \geq 2$, $B_\lambda$ is described and an associated invariant measure is constructed. The resulting system is shown to be isomorphic to a Bernoulli system. The corresponding orthogonal polynomials are considered in detail, and are shown to generalize Tchebycheff polynomials. For $-\frac{1}{4} < \lambda < 2$, and for $|\lambda| < \frac{1}{4}$ with $\lambda \in \mathbb{C}$, the nature of $B_\lambda$ as a subset of the complex plane is characterized and described. The relationship between $B$ and the theory of iterated mappings from a real interval into itself, is explained.
1. Introduction

In this paper we consider a family of invariant sets which are associated with a one-parameter family of quadratic maps from the complex plane \( \mathbb{C} \) into itself. The parameter is \( \lambda \), which may be real or complex, and the mapping \( T_\lambda : \mathbb{C} \rightarrow \mathbb{C} \) is defined by

\[
T_\lambda z = (\lambda - z)^2, \quad z \in \mathbb{C}.
\]

(1)

For \( \lambda \neq 0 \), \( T_\lambda \) is related to the mapping \( T^\nu \), where

\[
T^\nu z = 1 - \lambda z^2,
\]

(2)

which has been extensively studied in the context of iterated maps, see Feigenbaum [11], and also the book by Collet and Eckmann [9] and the references therein. The relationship is

\[
T^\nu = A^{-1} T_\lambda A,
\]

(3)

where \( A \) is the affine transformation

\[
A z = \lambda - \lambda z, \quad A^{-1} z = 1 - z/\lambda.
\]

(4)

\( T^\nu_\lambda \) is usually viewed as a mapping from the real interval \([-1,1]\) into itself, and then \( \lambda \) lies in the interval \([0,2]\). This is equivalent to looking at \( T_\lambda \) as a mapping from the interval \([0,2\lambda]\) into itself, which again corresponds to \( \lambda \in [0,2] \). In these cases notice that the critical values of \( \lambda \), at which occur such phenomena as the first appearance of \( k \)-cycles and the onset of
ergodic behavior, are the same for both mappings. Another equivalent transformation is $z \rightarrow z^2 - \lambda$.

The invariant set which we study will be denoted by $B$, and is among those first examined by Fatou [10] and Julia [16] in the context of an arbitrary rational transformation. With the notation

$$T_\lambda^0 z = z, \quad T_\lambda^{n+1} z = T_\lambda (T_\lambda^nz) \text{ for } n \in \{0,1,2,\ldots\}$$

$B_\lambda$ can be defined to be those points in the complex plane where the sequence $\{T_\lambda^nz\}$ is not normal. This definition is the starting point of the survey by Brolin [7]. However, we will adopt a different approach.

We begin with the set $\mathbb{B}_\lambda$ of formal objects $\{\lambda \pm (\lambda \pm \cdots \text{ad infinitum})\}$ where all possible sequences of plus and minus signs are included. Then we seek a corresponding subset $B_\lambda$ of the complex plane which possesses all of the properties which are suggested by $\mathbb{B}_\lambda$. To see what these properties are, and why they are of interest, let us pretend for the moment that $\mathbb{B}_\lambda$ makes computational sense, and describe briefly the results of some numerical experiments which we carried out at the inception.

For various sequences of, say, fifty plus and minus signs and for various values of $\lambda$ we calculated the numbers

$$\lambda \pm \sqrt{(\lambda \pm \cdots \pm \sqrt{\lambda}) \cdots} \uparrow 50 \text{ times} \uparrow$$

For complex numbers $z = re^{i\theta}$ with $-\pi < \theta \leq \pi$ we set

$$\sqrt{z} = \sqrt{r} e^{i\theta/2},$$

and

$$-\sqrt{z} = \begin{cases} \sqrt{r} e^{i(\theta/2 + \pi)} & \text{if } -\pi < \theta \leq 0, \\ \sqrt{r} e^{i(\theta/2 - \pi)} & \text{if } 0 < \theta \leq \pi. \end{cases}$$
This prescription corresponds to a negative axis branch cut. The results which we obtained are essentially those displayed in Figures 3 to 6 in Section 4. The calculated set appears to become the unit circle \( \{ z \in \mathbb{C} \mid |z| = 1 \} \) as \( \lambda \) approaches zero, whilst for \( \lambda = 2 \) it lies thickly on the real interval \([0,4]\). For \( 0 < \lambda < 2 \) more complicated figures were obtained: for \( \lambda = 0.7 \) the set looks something like the boundary of a holly leaf. For \( \lambda > 2 \) it was found to lie upon the real axis and to consist of distinct points.

Noting that when \( \lambda = 0 \) the unit circle is actually an invariant set for \( T_0 \), whilst for \( \lambda = 2 \) the real interval \([0,4]\) is an invariant set for \( T_2 \), it becomes clear that what we are tracking in the above calculations is a certain invariant set of \( T_\lambda \). This being the case, the subset of this set which lies on the real line must be an invariant set for the restriction of the mapping to the real line. Now, a \( k \)-cycle for \( T_\lambda : \mathbb{R} \to \mathbb{R} \) is an invariant set; and it appears that \( \mathbb{B}_\lambda \) moves gradually from the complex plane onto the real line, with increasing real \( \lambda \), and this movement is coincident with the occurrence of real \( k \)-cycles for higher and higher \( k \) values. Thus, it is natural to associate with a \( k \)-cycle of \( T_\lambda \) one of the members of \( \mathbb{B}_\lambda \) which has a recurring sequence of plus and minus signs with period \( k \). For example, one anticipates that

\[
\lambda + \sqrt{\lambda} + \sqrt[3]{\lambda} - \sqrt[3]{\lambda} + \sqrt[3]{\lambda} - \cdots
\]

(9)

is associated with a 2-cycle of \( T_\lambda \). When it first becomes real one expects that a real 2-cycle has first appeared for that value of \( \lambda \).

The set \( \mathbb{B}_\lambda \), to be associated directly with \( \mathbb{B}_\lambda \) will possess the kinds of properties indicated above. It thus provides insight into the behavior of the real map and into such phenomena as the cascades of bifurcations, the Feigenbaum numbers, and the onset of (nontrivial) ergodic properties. In the end \( \mathbb{B}_\lambda \) becomes a convenient notation and mnemonic for thinking about the detailed structure of \( \mathbb{B}_\lambda \).
Similar computations to the ones mentioned above have been made by Mandlebrot [19], who has produced some beautiful pictures of $B_\lambda$. He views $B_\lambda$ as an example of a fractal set, and also as an attractor for an appropriately defined (generalized) discrete dynamical system, based upon inverse mappings.

One reason why we first became interested in $B_\lambda$ was because it arose for $\lambda = 3$ in the context of the Diophantine Moment Problem (D.M.P.) [3]. This appeared in an attempt to predict the critical indices for Ising model lattice gases, basing the investigation upon the fact that certain related series seem to involve only integer coefficients. The Diophantine Moment Problem is this: classify as a function of the positive parameter $\lambda$ the positive measures $\mu$ such that

$$\mu_n = \int_0^\lambda x^n \, d\nu(x)$$

is an integer for each $n \in \{0, 1, 2, \ldots\}$. The problem was completely solved for $\lambda < 4$ and largely solved for $\lambda = 4$ (leading, incidently, to a novel resolution for a one-dimensional Ising model). For $\infty > \lambda > 6$ it was shown that there exists a transformation upon the generating function

$$G_\lambda(w) = \int_0^\lambda \frac{d\nu(x)}{1+wx}.$$  

(11)

which yields a new generating function $G_\lambda(w)$ with $\lambda < 6$. The measure which is associated with the latter also has the Diophantine property. However, the point which is important to us here is that the support of the measure for a fixed point of the transformation was associated with $B_\lambda$. Our present investigation provides results which may bear directly upon the D.M.P. and certain Ising models, and which also relate these areas with the topic of iterated maps of intervals.
Our study of $B_\lambda$ also involves other areas of interest in mathematical physics, namely invariant measures, isomorphisms of systems, and orthogonal polynomials. The motion of the points of $B_\lambda$ under the mapping $T_\lambda$, and the explicit construction of the corresponding invariant measure is of interest in its own right (see for example Lasota and Yorke [17]) and has been considered previously by Brolin [7]. Here we establish ergodic properties of the resulting system by means of an isomorphism of systems, in the sense of Billingsley [5]. Also we characterize the invariant measure in terms of an associated family of orthogonal polynomials. The possibility of so doing is unusual because of the singular nature of the measure. A direct connection between $B_\lambda$ and a Jacobi matrix is obtained, which allows one to envisage a physical system whose spectrum is $B_\lambda$.

We have so far mentioned our subject matter, motivations, and objectives. We now summarize what we achieve. In Section 2 we examine the case $2 < \lambda < \infty$, for which it is known that $B_\lambda$ lies entirely upon the real axis. Some of our approach may be seen as a melding of ideas of Fatou and Brolin. In §2.1 we succeed in a direct construction of $B_\lambda$ which displays its connection with $B_\lambda$ and shows the operation of $T_\lambda$ on $B_\lambda$ to be equivalent to that of the right-shift operator upon the set $\Omega$ of half-infinite Bernoulli sequences; we also discover a distance function which is natural to $B_\lambda$, yielding a simple demonstration that its Lebesgue measure is zero when $\lambda > 2$, and providing a new upper bound for its Hausdorff dimension. Finally in §2.1 we construct the measure upon $B_\lambda$, which is invariant and mixing under $T_\lambda$, by means of a special sequence of approximating measures. This measure is singular with respect to Lebesgue measure and has no purely atomic component. In §2.2 we establish an isomorphism of systems which relates the invariant to the uniform measure upon $\Omega$, and shows that the action of $T_\lambda$ upon $B_\lambda$ has entropy equal to $\ln 2$. In §2.3 the approxi-
In natural measures are related to a set of monic polynomials, orthogonal with respect to the invariant measure. A method for calculating all of these polynomials is provided, and certain identities which obtain among the coefficients of the associated three-term recurrence relations are discovered. Interrelations between the polynomials reflect the structure of $B_\lambda$. In particular, it is found possible to completely describe an infinite subsequence of Padé approximants to the generating function. It is also shown that the polynomials are none other than the Tchebycheff polynomials when $\lambda = 2$, and that they generalize the latter in a nontrivial way when $\lambda > 2$.

In Section 3 we consider the cases $-\frac{1}{4} < \lambda < 2$ and $|\lambda| < \frac{1}{4}$ with $\lambda \in \mathbb{C}$. In §3.1, for $-\frac{1}{4} < \lambda < 2$, $B$ is realized as the boundary of the image under a conformal mapping $F_\lambda$ of the exterior of the unit circle, with the property

$$T_\lambda \circ F_\lambda = F_\lambda \circ T_0$$

(12)

The possibility of such characterizations in general was known to Fatou: equation (12) is the Böttcher equation (see [10] §8) for $T_\lambda$ about the point $z = \infty$, which is both a fixed point for $T_\lambda$ and a critical point for its inverse, and it relates the action of $T_\lambda$ on $B_\lambda$ to that of $T_0$ on $B_0$, the unit circle.

We describe the construction of $F_\lambda$ with the aid of a specific sequence of analytic functions $\{f_n(z)\}$ which converge to it. The corresponding decreasing set of boundaries which we obtain, and which converges to $B_\lambda$, illustrates in a more specific and constructive manner than is usually possible, a general technique in the area of iterated rational functions. The explicit form of the results is a consequence of having the formulas $\lambda \pm \sqrt{z}$ for the two branches of the inverse of $T_\lambda$. A new type of approximation to $B_\lambda$ is provided by a second sequence of analytic functions $\{f_n^*(z)\}$, which are associated with an increasing set of boundaries, and which, in the appropriate sense, approach $B_\lambda$ from the "inside." Our construction turns out to be of especial interest: we have recently proved [4] that the sequence of tree-like boundaries in fact con-
verges to \( B_\lambda \) itself, for infinitely many values of \( \lambda \in (0,2) \). In these cases \( B_\lambda \) is a "dendrite" and the set \( \hat{C} - B_\lambda \) has only one component.

In §3.2, for \( |\lambda| < \frac{1}{4} \) with \( \lambda \in \mathbb{C} \), we relate \( B_\lambda \) to \( \hat{\mathbb{C}} \) in a similar manner to that in §2.1 for \( 2 < \lambda < \infty \). First, we restrict attention to the case \( 0 < \lambda < 0.2 \); in this situation \( f_\lambda(z) \) is defined for all \( z \in \mathbb{C} \) such that \( |z| = 1 \), and we show how the value of \( f_\lambda(e^{i\theta}) \) for given \( \theta \in [0,2\pi) \) can be identified with a member of \( \hat{\mathbb{C}} \) (this being defined using a positive axis branch cut). Second, we extend the results to the case \( \lambda \in L, L = \{ \lambda \in \mathbb{C} \mid |\lambda| < \frac{1}{4} \} \), by showing that \( f_\lambda(z) \) is both defined and analytic in \( \lambda \), for \( \lambda \in L \). Last, we show that the family of functions \( G_\lambda(\theta) = f_\lambda(e^{i\theta}) \), where \( \lambda \) belongs to a compact subset of \( L \), is uniformly Hölder continuous in \( \theta \).

Numerical results are illustrated in Section 4.

We thank D. Ruelle for telling us the references to Fatou, Julia, Brolin, and Mandelbrot. Our approach, the search for a meaning for the set \( \hat{\mathbb{C}} \), was discovered without these. We have preserved this path in the rewrite, and have signposted the classical results as we pass them. The reason is that, as well as providing a fresh viewpoint, we find the variety of new results summarized above.
§2. The case $2 \leq \lambda < \infty$

2.1 Construction of $B_{\lambda}$ and of an invariant measure

Throughout this section we assume $2 \leq \lambda < \infty$, and some the results are restricted to the case $\lambda > 2$. We begin with a direct construction of $B_{\lambda}$ which displays its connection with $\hat{\nu}$. This will be achieved with the aid of a new distance function which allows us to obtain a significant improvement on Brolin's estimate ([7], Theorem 12.2) of the Hausdorff dimension of the set.

Our construction shows the operation of $T_{\lambda}$ on $B_{\lambda}$ to be equivalent to that of the right-shift operator upon the set $\Omega$ of half-infinite Bernoulli sequences. Thus we obtain a concrete and fully worked out example of the type of correspondence apparently first introduced by Fatou ([10], §23) in the context of arbitrary rational transformations.

We will conclude this section with the construction of an invariant measure, supported on $B_{\lambda}$, by means of an approximating sequence of measures, different from the ones used by Brolin ([7], P. 126), with the advantage that they will later (section 2.3) turn out to be precisely the approximations to the measure which come from associated orthonal polynomials.

We define

$$a = \lambda + \frac{1}{2} + \sqrt{\frac{\lambda + 1}{2}} \quad \text{and} \quad I = [\lambda - \sqrt{a}, a].$$  \hspace{1cm} (1)

Notice that $a$ is the unique nonnegative real number satisfying $a = \lambda + \sqrt{a}$ and that $\lambda - \sqrt{a} \geq 0$ where the inequality is strict when $\lambda > 2$. For $n \in \{1, 2, 3, \cdots\}$ we define a function of $x \in I$ by

$$s_n(x) = \lambda + e_1 \sqrt{\lambda + e_2 \sqrt{\lambda + \cdots + e_{n-1} \sqrt{\lambda + e_n \sqrt{x}}}} \cdots)$$  \hspace{1cm} (2)

where each $e_i \in \{-1, 1\}$. Let $s_n$ denote the set of all functions expressible in the form (2). For $n=0$ we define $s_0(x) = x$ and $s_0$ contains only this function.
This Lemma shows that $\mathcal{B}_n$ is a well defined set of real valued functions on $I$.

**Lemma 1.** Let $2 \leq \lambda < \infty$ and $s_n(x) \in \mathcal{B}_n$. Then $s_n(x)$ is a real valued function of $x \in I$; in fact

$$s_n(x) \in I \text{ for all } x \in I$$ (3)

**Proof.** The Lemma is clearly true for $n=0$. Assume that it is true for all $n \in \{0,1,2,\ldots,N\}$ for some nonnegative integer $N$. Notice that for $s_{N+1} \in \mathcal{B}_{N+1}$ there exists $s_n \in \mathcal{B}_N$ and $e \in \{-1,+1\}$ such that

$$s_{N+1}(x) = \lambda + e \sqrt{s_n(x)}.$$ (4)

$s_n(x)$ is positive for all $x \in I$ by the inductive hypothesis, and so $s_{N+1}(x)$ is real valued. In fact

$$\lambda - \sqrt{a} \leq \lambda - \sqrt{s_n(x)} \leq s_{N+1}(x) \leq \lambda + \sqrt{s_n(x)} \leq \lambda + \sqrt{a} = a,$$ (5)

for all $x \in I$, and this proves (3) for $n = N + 1$. Q.E.D.

We introduce a measure $\mu$ on $I$ by defining

$$\mu(E) = \int_E \frac{dw}{\sqrt{w(2\lambda-w)}}$$ (6)

for all Lebesgue measurable subsets $E$ of $I$. Then $\mu$ is absolutely continuous with respect to Lebesgue measure on $I$ because $\int_I \frac{dw}{\sqrt{w(2\lambda-w)}} < \infty$. (Notice that this is true even when $\lambda = 2$). Also, Lebesgue measure is absolutely continuous with respect to $\mu$ on $I$ because $\int_E dx = \int_E \sqrt{w(2\lambda-w)} \, d\mu(w)$. A subset of $I$ has Lebesgue measure zero if and only if its measure with respect to $\mu$ is zero.
Corresponding to $\mu$ we have the distance function

$$d(x, y) = |F(x) - F(y)| \text{ for all } x, y \in I,$$

where we define

$$F(x) = \int_0^x \frac{dw}{\sqrt{w(2\lambda - w)}} \text{ for } x \in I$$  \hspace{1cm} (8)

Notice that the topology induced on $I$ by this metric is just the same as the usual topology which is associated with the distance function $|x-y|$.

**Lemma 2.** For given $2 \leq \lambda < \infty$, let $\rho = \frac{1}{2} \frac{\sqrt{\lambda + \sqrt{\lambda^2 - 2\lambda + \sqrt{\lambda^2 - 2\lambda + a^2}}}}{\sqrt{\lambda^2 - \lambda + \sqrt{\lambda^2 - 2\lambda + a^2}}}$. Then

$$d(\lambda + e\sqrt{2}, \lambda + e\sqrt{2}) \leq \rho d(x, y)$$

for all $x, y \in I$ and $e \in \{-1, 1\}$. When $\lambda > 2$, $0 < \rho < \frac{1}{2}$.

**Proof.** We use Cauchy's mean value theorem. Without loss of generality we assume $x < y$. We have

$$d(\lambda + e\sqrt{x}, \lambda + e\sqrt{y}) = \frac{|F(\lambda + e\sqrt{x}) - F(\lambda + e\sqrt{y})|}{|F(x) - F(y)|}$$

$$= \left| \frac{1}{2} \sqrt{\frac{e}{C}} \cdot \frac{F'(C)}{F'(C)} \right| \leq \frac{1}{2} \sqrt{\frac{2\lambda - C}{\lambda^2 - C}}$$

(10)

for some $C \in (x, y)$. Since $0 \leq \lambda \sqrt[a]{\sqrt{\lambda^2 - 2\lambda + a^2}} < x < C < y < a < 2\lambda < \lambda^2$ we discover that the right-hand-side is bounded above by $\rho = \frac{1}{2} \frac{\sqrt{\lambda + \sqrt{\lambda^2 - 2\lambda + a^2}}}{\sqrt{\lambda^2 - \lambda + \sqrt{\lambda^2 - 2\lambda + a^2}}} \leq \frac{1}{2}$. The latter inequality is in fact strict when $\lambda > 2$, and the Lemma follows. Q.E.D.
LEMMA 3. Given $2 \leq \lambda < \infty$

$$d(s_n(x), s_n(y)) \leq \rho^n d(x,y)$$  \hfill (11)

for all $x, y \in I$, and all $s_n \in \beta^n$, where $0 < \rho \leq \frac{1}{2}$.

Proof. The Lemma is immediately true for $n=0$ since $s_0(x) = x$. Assume that it is true for all $n \in \{0,1,2,\cdots,N\}$ for some nonnegative integer $N$. Let $s_{N+1} \in \beta^{N+1}$, so that (4) is true for some $s_N \in \beta^N$. Take $x$ to be $s_N(x)$ and $\lambda$ to be $s_N(y)$ in Lemma 2 yielding

$$d(s_{N+1}(x), s_{N+1}(y)) \leq \rho d(s_N(x), s_N(y))$$  \hfill (12)

for any $x, y \in I$. Now apply the inductive hypothesis to the right-hand-side of (12).

Q.E.D.

Let $\Omega$ denote the set of all semi-infinite sequences of numbers from \{-1,+1\}. Then $\omega \in \Omega$ if and only if it can be expressed

$$\omega = (e_1,e_2,e_3,\cdots)$$  \hfill (13)

where each $e_i$ is in \{-1,+1\}. In order to indicate the dependence of the function $s_n(x)$ in (2) upon $\omega$ we will write

$$s_n(x) = s_n(x;\omega).$$  \hfill (14)

The following theorem is related to a general result concerning the convergence of infinite chains of inverse branches of a rational map [7, Theorem 6.2].
THEOREM 1 Let $\omega \in \Omega$, $2 \leq \lambda < \infty$, and $x \in I$. Then $s(\omega) = \lim_{n \to \infty} s_n(x, \omega)$ exists, belongs to $I$, and is a constant independent of $x \in I$.

Proof. First observe that from Lemma 2

$$d(s_{n+1}(x; \omega), s_n(x, \omega)) = d(s_n(\lambda + e^{-\lambda} \sqrt{x}; \omega), s_n(x; \omega))$$

$$\leq \rho^n d(\lambda + e^{-\lambda} \sqrt{x}, x) \leq \rho^n c$$

for all $x \in I$, where both $\rho$ with $0 < \rho < \frac{1}{2}$ and

$$C = d(\lambda - \sqrt{a}, a)$$

are constants independent of $n$ and $x$. Hence, for any integer $p \geq 0$,

$$d(s_{n+p}(x; \omega), s_n(x; \omega)) \leq \rho^n (1 - \rho)^{-1} C$$

and we deduce that $\{s_n(x; \omega)\}_{n=0}^\infty$ is a Cauchy sequence in $I$ convergent to some number $s(\omega)$ in $I$. To see that $s(\omega)$ is independent of $x$ we suppose

$$\lim_{n \to \infty} s_n(x; \omega) = s(\omega) \text{ and } \lim_{n \to \infty} s_n(y; \omega) = s(\omega), \text{ for some } y \in I.$$ Then for any $n$

$$d(s(\omega), s(\omega)) \leq d(s(\omega), s_n(x; \omega)) + d(s(\omega), s_n(y; \omega)) + d(s_n(x; \omega), s_n(y; \omega)).$$

The first two terms on the right can be made arbitrarily small by choosing $n$ sufficiently large. The last term can be made arbitrarily small by virtue of Lemma 2. Q.E.D.

Let $B_\lambda$ denote the set of formal objects $\{\lambda + \sqrt{\lambda + \sqrt{\lambda + \cdots}}\}$ where all possible sequences of plus and minus signs are included. For $2 \leq \lambda < \infty$,
denote a subset of the real numbers by

\[ B_\lambda = \{ s(\omega) | \omega \in \Omega \}. \] (19)

Then Theorem 1 provides us with an identification mapping from \( B_\lambda \) onto \( B_\lambda \). We simply introduce the notation

\[ s(\omega) = \lambda + e_1 \sqrt{\lambda + e_2 \sqrt{\lambda + e_3 \sqrt{\cdots}}} \] (20)

for the limit provided by the theorem. Theorem 2 will tell us that the mapping is one-to-one when \( \lambda > 2 \), and in this case Theorem 1 provides us with an invertible mapping from \( \Omega \) into \( B_\lambda \) according to \( \omega \mapsto s(\omega) \). It also shows how the set \( B_\lambda \) can be calculated in practice.

Let us define \( T_\lambda : \mathbb{C} \to \mathbb{C} \) by

\[ T_\lambda z = (z-\lambda)^2, \text{ for } z \in \mathbb{C}. \] (21)

Let us also define \( T : \Omega \to \Omega \) to be the left shift operator; that is

\[ T(e_1, e_2, e_3, \cdots) = (e_2, e_3, e_4, \cdots) \] (22)

Notice that \( \Omega \) is an invariant set for \( T \). The next theorem relates the action of \( T_\lambda \) on \( B_\lambda \) to the action of \( T \) on \( \Omega \). It makes precise the type of correspondence described by Fatou [10, §23] for general rational transformations.
THEOREM 2. Let $2 \leq \lambda < \infty$. Then $B_\lambda$ is an invariant set for $T_\lambda$, and for any $\omega \in \Omega$,

$$T_\lambda s(\omega) = s(T_\omega).$$  \hfill (23)

If $\lambda > 2$, and if $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$, then $s(\omega_1) \neq s(\omega_2)$.

Proof. Since $T_\lambda : \mathbb{E} \rightarrow \mathbb{E}$ is continuous we have

$$T_\lambda s(\omega) = \lim_{n \to \infty} T_\lambda s_n(x;\omega)$$  \hfill (24)

for any $x \in I$ and any $\omega \in \Omega$. But for $n \geq 1$

$$T_\lambda s_n(x;\omega) = \left( (\lambda + e_1 \sqrt{\lambda + e_2 \sqrt{\cdots + e_n \sqrt{x}}}) - \lambda \right)^2$$

$$= \lambda + e_2 \sqrt{\lambda + e_3 \sqrt{\cdots + e_n \sqrt{x}}} = s_{n-1}(T_\omega;x)$$

whence

$$T_\lambda s(\omega) = \lim_{n \to \infty} s_{n-1}(T_\omega;x) = s(T_\omega)$$  \hfill (26)

This proves (23) and in particular that $T_\lambda : B_\lambda \rightarrow B_\lambda$.

Now suppose $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$, and without loss of generality suppose

$$\omega_1 = (e_1, e_2, \ldots, e_{n-1}, 1, e_{n+1}, e_{n+2}, \ldots),$$

$$\omega_2 = (e_1, e_2, \ldots, e_{n-1}, 1, e_{n+1}, e_{n+2}, \ldots).$$
\[ \omega_2 = (e_1, e_2, \ldots, e_{n-1}, -e_n, e_{n+1}, e_{n+2}, \ldots). \]  

(27)

Write \((e_{n+1}, e_{n+2}, \ldots) = \omega \) and \((e_{n+1}, e_{n+2}, \ldots) = \omega'. \)

Then
\[ T^{n-1}_\lambda s(\omega_1) = s(T^{n-1}_\lambda \omega_1) = \lambda + \sqrt{s(\omega)}, \]

and
\[ T^{n-1}_\lambda s(\omega_2) = s(T^{n-1}_\lambda \omega_2) = \lambda - \sqrt{s(\omega)}. \]  

(28)

Since both \(s(\omega)\) and \(s(\omega')\) both belong to \(I\) and \(0 \notin I\) when \(\lambda > 2\) we conclude that \(\omega_1 \neq \omega_2\) and \(\lambda > 2\) implies \(T^{n-1}_\lambda s(\omega_1) \neq T^{n-1}_\lambda s(\omega_2)\) for some \(n\), which in turn implies \(s(\omega_1) \neq s(\omega_2)\). Q.E.D.

We consider next what the set \(B_{\lambda} \) looks like. Let us define inductively real points \(x^{(n)}_k\) for \(n \in \{0, 1, 2, \ldots\}\) and \(k \in \{1, 2, \ldots, 2^{n+1}\}\) by
\[ x_1^{(0)} = \lambda - \sqrt{a}, \quad x_2^{(0)} = a, \]  

and for \(n \in \{1, 2, \ldots\}\) and \(k \in \{1, 2, \ldots, 2^{n-1}\}\)
\[ x_{2k-1}^{(n)} = \lambda - \sqrt{x_{2k-1}^{(n-1)}} \]

\[ x_{2k}^{(n)} = \lambda - \sqrt{x_{2k}^{(n-1)}} \]

\[ x_{2k-1+2^n}^{(n)} = \lambda + \sqrt{x_{2k-1}^{(n-1)}} \]

\[ x_{2k+2^n}^{(n)} = \lambda + \sqrt{x_{2k}^{(n-1)}} \]

For example, on taking \( n = k = 1 \) in (30) we obtain

\[ x_1^{(1)}(1) = \lambda - \sqrt{x_2^{(0)}} \]

\[ x_2^{(1)}(1) = \lambda - \sqrt{x_1^{(0)}} \]

\[ x_3^{(1)}(1) = \lambda + \sqrt{x_1^{(0)}} \]

\[ x_4^{(1)}(1) = \lambda + \sqrt{x_2^{(0)}} \]

For \( n \in \{0, 1, 2, \ldots\} \) we introduce the notation

\[ I^{(0)}(n) = \bigcup_{k=1}^{2^n} [x_{2k-1}^{(n)}, x_{2k}^{(n)}]. \]

Notice that \( I^{(0)} = I \).

**Lemma 4** Let \( 2 < \lambda < \infty, \omega \in \Omega, x \in I, \) and \( m \in \{0, 1, 2, \ldots\} \). Then

\[ s_n(x; \omega) \in I^{(m)} \] for all \( n \geq m \). In particular, \( s(\omega) \in I^{(m)} \).

**Proof.** First we prove that for all \( x \in I \) and \( \omega \in \Omega \)

\[ s_n(x; \omega) \in I^{(m)}. \]

This is achieved by induction. It is true for \( m = 0 \) by Lemma 1. Assume that
it is true for all \( m \in \{0, 1, 2, \ldots, N\} \) for some nonnegative integer \( N \). Observe that for some \( \tilde{\omega} \in \Omega \) either (i) \( s_{n+1}(x; \tilde{\omega}) = \lambda + \sqrt{s_N(x; \tilde{\omega})} \), or (ii) \( s_{n+1}(x; \tilde{\omega}) = \lambda - \sqrt{s_N(x; \tilde{\omega})} \). Applying the inductive hypothesis in case (i) we have

\[
s_N(x; \tilde{\omega}) \in \bigcup_{k=1}^{2^N} \left[ x_{2k-1}^{(N)}, x_{2k}^{(N)} \right]
\]

whence \( s_{n+1}(x; \tilde{\omega}) \in \bigcup_{k=1}^{2^N} \left[ x_{2k-1}^{(N+1)}, x_{2k}^{(N+1)} \right] \), and similarly in case (ii) we find \( s_{n+1}(x; \tilde{\omega}) \in \bigcup_{k=1}^{2^N} \left[ x_{2k-1}^{(N+1)}, x_{2k}^{(N+1)} \right] \). In both cases we have (33) with \( m = N + 1 \). This proves (33).

Now let \( n \geq m \) and notice that for some \( \tilde{\omega} \in \Omega \)

\[
s_n(x; \tilde{\omega}) = s_m(s_{n-m}(x; \tilde{\omega}); \tilde{\omega}). \quad (34)
\]

Noticing that \( s_{n-m}(x; \tilde{\omega}) \in I \) and applying (33) we obtain \( s_n(x; \tilde{\omega}) \in I^{(m)} \). Taking the limit as \( n \to \infty \) and using the compactness of \( I^{(m)} \) we obtain

\[
S(\tilde{\omega}) \in I^{(m)}. \quad \text{Q.E.D.}
\]

We will say that a set of points is a Cantor set if it is compact, non-denumerable, and contains no intervals. A set is perfect if every element is a limit point of other members of the set and the set contains all of its limit points. The following theorem summarizes the structure of \( B_\lambda \) when \( 2 < \lambda < \infty \).

Our proof that the Lebesgue measure is zero makes direct use of the distance function \( d(x, y) \), and is considerable simpler than other approaches - see for example [7, Theorem 10.2]. It also prepares the way for the calculation of the Hausdorff dimension.

**THEOREM 3.** For \( 2 < \lambda < \infty \), \( B_\lambda \) is a Cantor set with Lebesgue measure for zero. For \( 2 \leq \lambda < \infty \), \( B_\lambda \) is compact and perfect.
Proof. That $B_{\lambda}$ is uncountable follows from the fact that $\Omega$ is uncountable and the mapping $S: \Omega \to B_{\lambda}$ is one-to-one, when $\lambda > 2$.

By virtue of Lemma 4 the set $I^{(m)}$ provides a covering for $B_{\lambda}$. Let $n \in \{1, 2, 3, \cdots \}$. Then using (30) and exploiting the symmetry of the set

$$\{x_{(n)}^{(n)}\}_{k=1}^{2^{n+1}}$$

about the point $\lambda$ we have

$$\mu_1(n) = \sum_{k=1}^{2^{n-1}} d(x^{(n)}_{2k-1}, x^{(n)}_{2k})$$

$$= 2^{n-1} \sum_{k=1}^{2^{n-1}} d(x^{(n)}_{2k-1}, x^{(n)}_{2k})$$

$$= 2^{n-1} \sum_{k=1}^{2^{n-1}} d(\lambda + \sqrt{x^{(n-1)}_{2k-1}}, \lambda + \sqrt{x^{(n-1)}_{2k}})$$

$$\leq (2\rho) \sum_{k=1}^{2^{n-1}} d(x^{(n-1)}_{2k-1}, x^{(n-1)}_{2k})$$

$$= (2\rho) \mu_1(n-1),$$

(35)

where the inequality comes from Lemma 2 and $0 < 2\rho < 1$ because $\lambda > 2$. Using induction in this case we conclude that

$$\mu_1(n) \leq (2\rho)^n \mu_1(0)$$

(36)

and hence that $\mu_{B_{\lambda}} = 0$. Thus when $\lambda > 2$ the Lebesgue measure of $B_{\lambda}$ must be zero because Lebesgue measure is absolutely continuous with respect to $\mu$.

Next we prove that $B_{\lambda}$ is compact for $2 \leq \lambda < \infty$. Clearly $B_{\lambda}$ is bounded, so it suffices to prove that it is closed. Let $\{S(\omega_n)\}_{n=1}^{\infty}$ denote a Cauchy sequence in $B_{\lambda}$, convergent to $x^* \in \mathbb{R}$. We must show that $x^* \in B_{\lambda}$. But the sequence $\{\omega_n\}_{n=1}^{\infty}$ in $\Omega$ contains an infinite subsequence,
also denoted by \( \{\omega_n^{\infty}\}_{n=1}^{\infty} \), which is convergent to some \( \omega^* \in \Omega \). By convergence here we mean that given any positive integer \( N \) there exists an integer \( M_N \) such that the first \( N \) components of \( \omega_n \) and \( \omega^* \) are the same for all \( n \geq M_N \). But now it readily follows that \( \{s(\omega_n^{\infty})\}_{n=1}^{\infty} \) converges to \( s(\omega^*) \) whence \( X^* = s(\omega^*) \in B_{\lambda} \) as desired. In a similar manner, the fact that \( \Omega \) is perfect implies that property for \( B_{\lambda} \).

**Theorem 4.** For \( \lambda > 2 \) the Hausdorff dimension of \( B_{\lambda} \) is bounded above by the number

\[
\frac{\ln 2}{\ln \left( \frac{2}{\lambda^2 - 2\lambda + a} \right)}
\]

**Proof** This follows the same lines as [6, Theorem 12.2], except that here we use the distance function \( d(x,y) \) in place of \( |x-y| \), these being equivalent metrics when \( \lambda > 2 \). We consider the family of coverings \( \{x_{2^{k-1}}, x_{2^k}\}_{k=1}^{n} \) for \( n \in \{1,2,3,\ldots\} \), and let

\[
H_n(\alpha) = \frac{2^n}{\sum_{k=1}^{n} (d(x_{2^{k-1}}, x_{2^k}))^\alpha} \text{ for } 0 < \alpha < \infty. \tag{38}
\]

Then

\[
H_{n+1}(\alpha) \leq (2\rho^\alpha) H_n(\alpha), \tag{39}
\]

from which it follows that \( \lim_{n \to \infty} H_n(\alpha) \) will be finite if \( 2\rho^\alpha < 1 \), which corresponds to \( \alpha > (\ln \frac{1}{2})/\ln \rho \). This implies that \( (\ln \frac{1}{2})/\ln \rho \) is an upper bound to the Hausdorff dimension of \( B_{\lambda} \). Q.E.D.

Brolin has given the following upper bounds for the Hausdorff dimension of \( B_{\lambda} \):
and

$$\frac{\ln 2}{\ln 2(2\lambda - a)^{1/2}}$$ valid for $2 + \frac{a}{2}$,

(40)

and

$$\frac{\ln 2}{\exp\{-60(\ln(2\lambda - a)^{1/2})^2\}} + \ln 2$$ valid for $2 < \lambda \leq b$.

(41)

Our bound (37) improves over both of these, where they apply. We note that for $\lambda = 5$, (37) yields the upper bound 0.564 whilst (40) gives 0.636. Thus our bound is good enough, at $\lambda = 5$, to distinguish $B^\lambda$ from the classical ternary set of Cantor, whose Hausdorff dimension is $\ln 2/\ln 3 = 0.631$, [15].

Our next objective is the construction of an invariant measure for $T^\lambda$, supported upon $B^\lambda$. Brolin also has given a construction for the measure, but using a different sequence of approximating measures. Ours will turn out (Section 2.3) to be connected directly with the associated orthogonal polynomials.

Let

$$K^n = \{s(\lambda)| s \in \mathcal{S}_n \} \text{ for } n \in \{1,2,3,\ldots\}. \quad (42)$$

Then $K^n$ contains $2^n$ distinct points which we denote by $\{Y_j^{(n)}\}_{j=1}^{2^n}$, these being ordered so that

$$Y_1^{(n)} < Y_2^{(n)} < \cdots < Y_{2^n}^{(n)}. \quad (43)$$

Recall the set of points $\{x_j^{(n-1)}\}_{j=1}^{2^n}$ which were introduced in connection with Lemma 4. It is not hard to see that

$$[Y_{2k-1}^{(n)}, Y_{2k}^{(n)}] < [x_{2k-1}^{(n-1)}, x_{2k}^{(n)}], \quad (44)$$

which fact we will find useful in what follows.

We define a family of distribution functions by
\[ \sigma_n(x) = \frac{1}{2^n} \sum_{y \in K_n} \Theta(x-y), \ x \in \mathbb{R}, \ n \in \{1, 2, 3, \ldots\}, \]  
(45)

where

\[ \Theta(x) = \begin{cases} 
0 & \text{when } x \leq 0, \\
1 & \text{when } x > 0. 
\end{cases} \]  
(46)

These distributions can be reexpressed

\[ \sigma_n(x) = \frac{1}{2^n} \left[ \text{Number of members of } K_n \text{ which are less than } x \right] \]

\[ = \left[ \text{Proportion of members of } K_n \text{ which are less than } x \right]. \]  
(47)

**Lemma 5** For \(2 \leq \lambda < \infty\) the sequence \(\{\sigma_n(x)\}_{n=1}^{\infty}\) converges to a continuous distribution function \(\sigma(x)\), uniformly for \(x \in \mathbb{R}\).

**Proof.** We have

\[
|\sigma_n(x) - \sigma_{n+1}(x)| = \frac{1}{2^n} \left| \sum_{y \in K_n} \Theta(x-y) - \frac{1}{2} \sum_{z \in K_{n+1}} \Theta(x-z) \right| \leq \frac{1}{2^n} \left| \sum_{s \in \mathbb{Z}_n} q(x,s) \right| 
\]

(48)

where

\[ q(x,s) = |\Theta(x-s(\lambda)) - \frac{1}{2} \Theta(x-s(\lambda + \sqrt{\lambda})) - \frac{1}{2} \Theta(x-s(\lambda - \sqrt{\lambda}))|. \]  
(49)

The two numbers \(s(\lambda \pm \sqrt{\lambda})\) are the endpoints of an interval \([Y_{2k-1}^{(n+1)}, Y_{2k}^{(n+1)}]\) for some \(k\), and we find
\[ q(x,s) = \begin{cases} 
0 \text{ when } x \in (-\infty, \gamma_{2k+1}^{(n+1)}) \cup (\gamma_{2k}^{(n+1)}, \infty), \\
1/2 \text{ when } x \in (\gamma_{2k-1}^{(n+1)}, \gamma_{2k}^{(n+1)}]. 
\end{cases} \tag{50} \]

Since the intervals \((\gamma_{2k-1}^{(n+1)}, \gamma_{2k}^{(n+1)})\) for \(k \in \{1, 2, \ldots, 2^n\}\) are disjoint and since each \(s \in \mathcal{F}_n\) leads to a different interval, we conclude from (48) that

\[ |\sigma_n(x) - \sigma_{n+1}(x)| \leq \frac{1}{2^{n+1}} \text{ for all } x \in \mathbb{R}. \tag{51} \]

Hence \(\{\sigma_n(x)\}\) is a Cauchy sequence which converges to some function \(\sigma(x)\), uniformly for \(x \in \mathbb{R}\). Clearly \(\sigma(x)\) is monotone nondecreasing and

\[ \sigma(x) = \begin{cases} 
0 \text{ for } x \leq \lambda - \sqrt{a}, \\
1 \text{ for } x \geq a. 
\end{cases} \tag{52} \]

To prove the continuity of \(\sigma\) at \(x\) we observe that

\[ |\sigma(x) - \sigma(y)| \leq |\sigma(x) - \sigma_n(x)| + |\sigma_n(x) - \sigma_n(y)| + |\sigma_n(y) - \sigma(y)|. \tag{53} \]

Let \(\varepsilon > 0\) be given and fix \(n\) so that simultaneously \(|\sigma(x) - \sigma_n(x)| < \varepsilon/4\) for all \(x \in \mathbb{R}\) and \(1/2^n < \varepsilon/2\). Without loss of generality let \(x < y\) and notice that

\[ \sigma_n(x) - \sigma_n(y) = \frac{1}{2^n} \sum_{z \in K_n} \{\Theta(x-z) - \Theta(y-z)\}. \tag{54} \]

But

\[ \Theta(x-z) - \Theta(y-z) = \begin{cases} 
0 \text{ when } z \in (-\infty, x) \cup (y, \infty), \\
1 \text{ when } z \in (x, y]. 
\end{cases} \tag{55} \]
so if we choose $\delta > 0$ so small that the interval $(x-\delta, x+\delta)$ contains at most one point of the set $K_n$, then

\[ |\sigma_n(x) - \sigma_n(y)| \leq \frac{1}{2^n} < \frac{\varepsilon}{2} \text{ for all } y \in (x-\delta, x+\delta). \]  \hspace{1cm} (56)

Thus

\[ |\sigma(x) - \sigma(y)| < \varepsilon \text{ for all } y \in (x-\delta, x+\delta). \]  \hspace{1cm} Q.E.D.  \hspace{1cm} (57)

**REMARK:** The distribution function $\sigma(x)$ is associated with the set $B_\lambda$ in the following way. $\frac{d\sigma(x)}{dx}$ exists and equals zero for all $x \not\in B_\lambda$. Thus $\sigma(x)$ does not increase for $x$ outside the set $B_\lambda$.

Previously we showed $B_\lambda$ to be an invariant set for $T_\lambda$. We now show how $\sigma(x)$ provides an invariant measure for $T_\lambda$. Since $\sigma(x)$ is continuous and monotone increasing we have from [20] (Ch. 12 Prop. 12) that there is a unique Borel measure, which for economy of notation we denote by $\sigma$, such that

\[ \sigma(c, d] = \sigma(d) - \sigma(c) \]  \hspace{1cm} (58)

for all $c < d$ in $\mathbb{R}$. We denote the corresponding Borel measurable subsets of $\mathbb{R}$ by $B$ so that $(\mathbb{R}, B, \sigma)$ is a measure space. For $E \in B$ we will use the notation

\[ \sigma_E = \int_E d\sigma(x). \]  \hspace{1cm} (59)

Notice that $\sigma_I = 1$ and that we could if wished restrict attention to $(I, B, \sigma)$ rather than $(\mathbb{R}, B, \sigma)$. 


THEOREM 5 For $\lambda \geq 2$ the measure $\sigma$ is invariant under $T_\lambda$. That is, for each $E \in B$,

$$\sigma E = \sigma T^{-1}_\lambda E. \quad (60)$$

In particular, for all measurable functions $f$,

$$\int_E f(x) \, d\sigma(x) = \int_{T^{-1}_\lambda E} f(Tx) \, d\sigma(x). \quad (61)$$

Proof. We begin by proving that $\sigma$ obeys (60) when $E = (c, d]$ is an interval. In this case from Lemma 5 we have

$$\sigma E = \lim_{n \to \infty} \int_{K_n \cap E} \, d\sigma(x) = \lim_{n \to \infty} \sigma_{n E}. \quad (62)$$

Here $\sigma_{nE} = \frac{1}{2^n}$ [Number of elements in $K_n \cap E$]. To each element $y$ of $K_n \cap E$ there corresponds exactly two elements $\lambda - \sqrt{y}$ and $\lambda + \sqrt{y}$ of $K_{n+1} \cap T^{-1}_\lambda E$, and these two elements correspond to no other element of $K_n \cap E$ than $y$. Accordingly

$$\sigma_{nE} = \frac{1}{2^{n+1}} \left\{ \text{Number of elements of } K_{n+1} \cap T^{-1}_\lambda E \right\} = \sigma_{n+1} T^{-1}_\lambda E, \quad (63)$$

and from (62)

$$\sigma E = \lim_{n \to \infty} \sigma_{n+1} T^{-1}_\lambda E = \sigma T^{-1}_\lambda E. \quad (64)$$
This proves (60) in the case where \( E \) is an interval. To extend the result to an arbitrary member of \( B \) we notice that a new measure on \( B \) can be defined by

\[
\bar{\mu}_E = \mu T^{-1} E
\]  

(65)

Then since \( \mu \) and \( \bar{\mu} \) agree on all intervals their extensions to all of \( B \) must agree, which is precisely (60).

It suffices to prove (61) in the case where \( f(x) = I_E(x) \) is the characteristic function of the set \( E \subseteq B \) for then it is true also for simple functions (which are finite linear combinations of characteristic functions) and finally, by closure, it is true for all measurable functions. Observing that \( T^{-1}_x(x) = I_E(T_x x) \), we have

\[
\int I_E(x) \, d\sigma(x) = \sigma E = \sigma T^{-1}_x E = \int I_{T_x^{-1}_x} (x) \, d\sigma(x) = \int I_E(T_x x) \, d\sigma(x). \quad (66)
\]

Q.E.D.
§2.2 Isomorphisms and Ergodic properties.

Let $(\Omega, F, \nu)$ denote a probability measure space. That is, $\Omega$ is an underlying set of points, $F$ is a sigma-algebra of subsets of $\Omega$ such that $\Omega \in F$, and $\nu$ is a countably additive measure such that $\nu(\Omega) = 1$. Let $T$ be a transformation of $\Omega$ into itself which preserves the measure $\nu$. Then we refer to $(\Omega, F, \nu, T)$ as a system. For example, Theorem 5 tells us that $(\mathbb{R}, B, \sigma, T)$ is a system.

In order to describe more fully than heretofore the nature of the action of $T$ on $B$, we need the concept of isomorphic systems. We recall the definition given by Billingsley [5] of an isomorphism between two systems $(\Omega, F, \nu, T)$ and $(\bar{\Omega}, F, \bar{\nu}, \bar{T})$.

Definition. Suppose there exist sets $\Omega_0$ in $F$ and $\bar{\Omega}_0$ in $\bar{F}$, each of measure 1, and a mapping $\phi$ of $\Omega_0$ onto $\bar{\Omega}_0$, with the following properties.

(i) The mapping $\phi$ is one-to-one.

(ii) If $A \subset \Omega_0$ and $A = \phi A$, then $A \subset F$ if and only if $\bar{A} \subset \bar{F}$ (i.e. $\phi$ is bimeasurable), in which case $\nu A = \bar{\nu} A$

(iii) We have $\Omega_0 \subset T^{-1} \Omega_0$, $\bar{\Omega}_0 \subset \bar{T}^{-1} \bar{\Omega}_0$, and

$$\phi T \omega = \bar{T} \phi \omega \quad \text{for all } \omega \in \Omega_0. \quad (1)$$

In this case we say that $(\Omega, F, \nu, T)$ and $(\bar{\Omega}, F, \bar{\nu}, \bar{T})$ are isomorphic.

There are two ways in which isomorphisms between systems can be used. They can be used to calculate the entropy of a given system and to determine whether or not the system is mixing or ergodic, because such properties are invariant under isomorphism. They can also be used to provide a conceptually simpler or more familiar replacement for an apparently alien system. We will exploit both of these ideas in what follows.
We define the measure space $(B_\lambda, \overline{B}, \overline{\sigma})$ to be the projection of $(\mathbb{R}, B, \sigma)$ onto $B_\lambda$ as follows. We define $\overline{B}$ to be the set of all subsets $E$ of $\mathbb{R}$ such that $E = E \cap B_\lambda$ for some $E \in B$. That is

$$\overline{B} = \{E = E \cap B_\lambda | E \in B\}. \quad (2)$$

For $\overline{E} \in \overline{B}$ with $\overline{E} = E \cap B_\lambda$ and $E \in B$ we define the measure $\overline{\sigma}$ by

$$\overline{\sigma}E \equiv \sigma E. \quad (3)$$

It is readily verified that $\overline{\sigma}$ is well defined, for if $\overline{E} = E_1 \cap B_\lambda = E_2 \cap B_\lambda$ then $\overline{\sigma}E_1 = \sigma(E_1 \cap B_\lambda) = \sigma(E_2 \cap B_\lambda) = \sigma(E_2 \cap B_\lambda) = \sigma(E_1 \cap B_\lambda) = \sigma(E_1 \cap B_\lambda)$. The superscript $c$ means the complement is taken. It is straightforward to prove that $(B_\lambda, \overline{B}, \overline{\sigma})$ constitutes a measure space and that $\overline{\sigma}$ is invariant under $T_\lambda$. As a simple example of isomorphism we have the following result.

**Lemma 6.** For $2 \leq \lambda < \infty$. The two systems $(\mathbb{R}, B, \nu, T_\lambda)$ and $(B_\lambda, \overline{B}, \overline{\sigma}, T_\lambda)$ are isomorphic.

**Proof.** We simply apply the definition of isomorphism taking $(\mathbb{R}, B, \sigma, T_\lambda)$ and $(\overline{\sigma}, \overline{B}, \overline{\nu}, \overline{T})$ to be $(\mathbb{R}, B, \sigma, T_\lambda)$ and $(\overline{\sigma}, \overline{B}, \overline{\nu}, \overline{T})$. Then we set $\overline{\Omega}_0 = B_\lambda = \overline{\Omega}_0$ and take $\phi$ to be the identity map. Q.E.D.

Now let $\Omega$ be as defined earlier, just above equation (13) in §2.1. We define the cylinder subsets of $\Omega$ by
where \( \mathbf{c} \) is given for \( \lambda \in \{0,1\} \) is given for \( \lambda \in \{n,n+1,\ldots,n+k\} \). We use the notation \( \mathcal{C} \) for the smallest sigma-algebra which contains all of the cylinder subsets. Then there exists a unique countably additive measure \( \xi \) on \( \mathcal{C} \) such that

\[
\xi(\mathbf{c}(i_n,i_{n+1},\ldots,i_{n+k})) = \frac{1}{2^{k+1}}
\]

for \( k \in \{0,1,2,\ldots\} \). Here let \( T \) denote the left shift operator on \( \Omega \) as defined in (22). Then it is well known that \((\Omega,\mathcal{C},\xi,T)\) is a system; in particular \( \xi \) is invariant under \( T \). This system is discussed at some length by Billingsley [5]. It is known to be ergodic, mixing, and to have entropy \( +\ln 2 \). Moreover, it is known to be isomorphic to the diadic transformation on the interval \([0,1)\) (Borel subsets, Lebesgue measure) defined by \( T_x = 2x \mod 1 \). Our main result, however, is contained in Theorem 5, which relates the \( \lambda \)-dependent system \((B_\lambda,\mathcal{B},\xi,\lambda T)\) to the \( \lambda \)-independent system 

\((\Omega,\mathcal{C},\xi,T)\).

Before stating the next theorem, we recall some definitions. Let \((\Omega,\mathcal{F},\nu,T)\) be a system. Then \( T \) is ergodic means that whenever \( A \in \mathcal{F} \) is invariant under \( T \), (i.e. \( A = T^{-1}A \)), then \( \nu(A) = 0 \), or 1. The Ergodic Theorem states that in this case, if \( f \) is integrable, \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k \omega) \) exists and equals \( \int f \, d\nu \), for almost all \( \omega \in \Omega \). \( T \) is mixing means

\( \lim_{n \to \infty} \nu(\mathcal{A} T^{-n} \mathcal{B}) = \nu(A) \nu(B) \) whenever \( A,B \in \mathcal{F} \). The entropy of a finite field \( G \) is defined by \( H(G) = -\sum \nu(A) \ln \nu(A) \), where the summation extends over the atoms of \( A \). The entropy of a finite field \( G \) relative to \( T \) is \( h(G,T) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^{-k} G \). (The atoms of \( \sum_{k=0}^{n-1} T^{-k} G \) are obtained by intersecting those of \( \sum_{k=0}^{n-1} T^{-k} G \), \( k \in \{0,1,2,\ldots,n-1\} \)). Finally, the entropy of \( T \) is \( h(T) = \sup h(G,T) \),
where the supremum extends over all finite subfields \( G \) of \( F \). Kolmogorov showed that the entropy of a system is invariant under isomorphism.

**THEOREM 6.** For \( 2 < \lambda < \infty \) the systems \((\mathbb{R}, B, \sigma, T_\lambda)\), \((B_\lambda, \overline{B}, \overline{\sigma}, T_\lambda)\), \((\Omega, C, \xi, T)\), and \(([0,1], \text{Borel subsets}, \text{Lebesgue measure}, \text{diadic transformation})\) are all isomorphic. In particular, each system is ergodic, mixing, and has entropy equal to \( \ln 2 \).

Proof. Once we have established that \((B_\lambda, \overline{B}, \overline{\sigma}, T_\lambda)\) and \((\Omega, C, \xi, T)\) are isomorphic systems the rest of the theorem follows at once from the previously mentioned isomorphisms, (isomorphism is transitive), together with the fact that the ergodic property, the entropy, and the mixing property are all invariant under isomorphism. The stated properties are well known for \((\Omega, C, \xi, T)\).

We will prove that an isomorphism between \((B_\lambda, \overline{B}, \overline{\sigma}, T_\lambda)\) and \((\Omega, C, \xi, T)\) is furnished by the mapping \( S: \Omega \rightarrow B_\lambda \) which was introduced following equation (20). We have already shown this mapping to be invertible when \( \lambda > 2 \) and we will use the notation \( S^{-1}: B_\lambda \rightarrow \Omega \) for its inverse. We clearly satisfy part (i) if the definition of isomorphism. We next prove (iii). To achieve this we must make an implication in each of two directions. First we show that

(ii)(a) If \( A \in C \) then \( S(A) \in \overline{B} \) and \( \xi A = \overline{\sigma}(S(A)) \).

It suffices to prove this assertion when \( A = c(i_k) \) is a thin cylinder subset of \( C \), for each positive integer \( K \), because these cylinders generate the whole of \( C \). But \( S(c(i_k)) \) is the set of points in \( B_\lambda \) which take the form

\[
\lambda \pm \sqrt{\lambda} \pm \sqrt{\lambda} \pm \cdots \pm i_k \sqrt{\lambda} \pm \sqrt{\lambda} \pm \cdots
\]

where only \( i_k \in \{-1, 1\} \) is fixed. This is just the set of all points of the form
\[ S(\lambda + i_k \sqrt{\lambda} - \sqrt{a}) \text{ and } S(\lambda + i_k \sqrt{a}) \quad (8) \]

for some \( S \in \mathcal{K}_{K-1} \). That is

\[ S(c(i_k)) = \{ u, I(S) \} \cap B_{\lambda} \quad (9) \]

The countable union of intervals here is a member \( E \) of \( B_{\lambda} \) so

\[ S(c(i_k)) = E \cap B_{\lambda} = \overline{E} \in B_{\lambda} \quad (10) \]

To complete the proof of (ii) (a) we must show that

\[ \xi c(i_k) = \overline{E} \quad (11) \]

On the one hand we have \( \xi c(i_k) = \frac{1}{2} \) from (5). On the other hand

\[ \overline{E} = \text{Lim}_{n \to \infty} \frac{1}{2^n} \{ \text{# of points of } K_n \text{ that lie in } E \} \]

\[ = \text{Lim}_{n \to \infty} \frac{1}{2^n} |K_n \cap E| \quad (12) \]

But \( \frac{1}{2^n} |K_n \cap E| = \frac{1}{2^n} |K_n \cap \{ u, I(S) \} | = \frac{1}{2} \) for all \( n > K \). This completes
the proof of (ii)(a). We must now prove that

(ii)(b) If $\bar{E}$ is in $\bar{B}$ then $S^{-1}(\bar{E}) \in C$ and $\bar{S}(\bar{E}) = \bar{S}^{-1}(\bar{E})$.

It suffices to prove this when

$$\bar{E} = [c, d] \cap B_\lambda$$

where $[c, d]$ is an arbitrary real closed interval. In this case $\bar{E}$ is compact so it has a smallest member, and a largest member, which can be written $S(\omega_1)$ and $S(\omega_2)$, respectively for some $\omega_1$ and $\omega_2$ in $\Omega$. Let

$$C_n = \{ \omega \in \Omega | S(\omega_1) < S_n(\omega, x) < S(\omega_2) \text{ for all } x \in B_\lambda \}.$$  \hspace{1cm} (14)

Then it is readily shown that

$$C_n \subset C_{n+1} \subset C_{n+2} \subset \cdots$$

Slightly less obvious is the identity

$$S^{-1}(\bar{E}) = \bigcup_{n=1}^{\infty} C_n \cup \{ \omega_1, \omega_2 \}.$$  \hspace{1cm} (16)

We show that this is indeed true. Suppose $\omega$ belongs to the right-hand-side of (16) but that $\omega \neq \omega_1$ and $\omega \neq \omega_2$, for in these latter cases it is immediate that $S(\omega) \in \bar{E}$. Then $\omega \in C_{n+k}$ for some finite integer $n$ and all nonnegative integers $k$. Hence $S(\omega_1) < S_{n+k}(\omega, x) < S(\omega_2)$ for all $k$ and all $x \in B_\lambda$ which implies $S(\omega_1) < S_n(\omega, x) < S(\omega_2)$. Hence $\omega \in S^{-1}(\bar{E})$. Conversely, suppose $\omega \in S^{-1}(\bar{E})$. Then $S(\omega) \in [S(\omega_1), S(\omega_2)]$. If $\omega = \omega_1$ or $\omega = \omega_2$ then $\omega$ belongs to the right-hand-side of (16) as desired. So assuming $\omega \neq \omega_1$ and $\omega \neq \omega_2$ we have $S(\omega_1) < S(\omega) < S(\omega_2)$. Now choose $N$ so large that for all $n \geq N$ we have

$$|S_n(\omega, x) - S(\omega)| < \min\{ |S(\omega) - S(\omega_1)|, |S(\omega) - S(\omega_2)| \}.$$  \hspace{1cm} (17)
for all \( x \in B^\lambda \). Then for all \( n > N \) we have that \( \omega C_n \), whence it belongs to the right-hand-side of (16).

Now that (16) is established we have that \( S^{-1}(E) \) is a countable union of cylinder sets because this is true of both \( \omega_1 \) and \( \omega_2 \), and each \( C_n \) is itself clearly a cylinder set. Hence \( S^{-1}(E) \in \mathcal{C} \) which is what we wanted. It remains for us to show that

\[
\tilde{\sigma}(E) = \xi S^{-1}(E). \tag{18}
\]

Here

\[
\tilde{\mu}(E) = \lim_{n \to \infty} \frac{1}{2^n} \left| K_n \cap [S(\omega_1), S(\omega_2)] \right| \tag{19}
\]

\[
= \lim_{n \to \infty} \frac{1}{2^n} \left[ \text{number of distinct first } n \text{ components possible for } \omega \in \Omega \text{ such that } S(\omega_1) < S_n(\omega, \lambda) < S(\omega_2) \right]
\]

On the other hand

\[
\xi S^{-1}(E) = \xi \cup C_n = \lim_{n=1}^{n \to \infty} \xi C_n \quad \text{(by virtue of (15))} \tag{20}
\]

\[
= \lim_{n \to \infty} \xi \{ \omega \in \Omega | S(\omega_1) < S_n(\omega;x) < S(\omega_2) \text{ for all } x \in B^\lambda \}
\]

\[
= \lim_{n \to \infty} \frac{1}{2^n} \left\{ \text{number of distinct first } n \text{ components possible for } \omega \in \Omega \text{ such that } S(\omega_1) < S_n(\omega;x) < S(\omega_2) \text{ for all } x \in B^\lambda \right\}
\]

But since each of the intervals \( \{ S_n(\omega;x) | x \in I \} \) is disjoint from all of the others, each interval contains exactly one point of the form \( S_n(\omega, \lambda) \) and conversely each such point is contained in only one interval. Hence the numbers
inside the curly brackets in (19) and (20) differ by at most two, and we have
the desired equality between (19) and (20) in the limit as \( n \rightarrow \infty \). This completes
the proof of (ii)(b).

Finally we must prove that

\[
(iii) \quad B_\lambda \subset T^{-1}_\lambda B_\lambda, \quad \Omega \subset T^{-1}_\lambda \Omega, \text{ and } S(T\omega) = T_\lambda(S(\omega)) \tag{21}
\]

But these were proved in the previous section; in particular (21) here is
(23) there. Q.E.D.

**REMARK** \((B, \bar{B}, \Omega, T_\lambda)\) is also isomorphic to the transformation \( T_0z = z^2 \) on
the unit circle with circular Lebesgue measure. This is true because the
latter system is isomorphic to the Diadic transformation on the unit interval,
see Billingsley [5] (Chapter 2). In this way we obtain a connection between
the system for \( \lambda > 2 \) and the system which exists for \( \lambda = 0 \).

There exists a natural ordering on the members of \( \Omega \). In the sequence
for a member of \( \Omega \) one replaces \(-1\) by \(0\) wherever it occurs; \( \Omega \) is then naturally
ordered by treating the resulting diadic sequences just as though they were
binary representations of real numbers, but with the distinction that any
binary expansion which has a tail of type form \( \cdots 011111 \cdots \cdots \)
is not to be identified with the same number whose tail has been altered to
\( \cdots 100000 \cdots \). Rather, we will say that the latter is strictly
larger than the former. This ordering of \( \Omega \) provides for \( \lambda > 2 \) an ordering on
\( B_\lambda \). We will write \( \omega_1 \prec \omega_2 \) to mean that \( \omega_1 \in \Omega \) is less than \( \omega_2 \in \Omega \) in this
ordering.

There exists a second ordering upon \( \omega \) and \( B_\lambda \). One can order the members
of \( \Omega \) according to the ordering of \( B_\lambda \) treated as a subset of the real line.
We will write $\omega_1 < \omega_2$ to mean that $\omega_1 \in \Omega$ is less than $\omega_2 \in \Omega$ in this ordering.

The connection between the two different orderings is provided by an invertible mapping $\Phi: \Omega \rightarrow \Omega$. For $\omega = (e_1, e_2, e_3, \cdots) \in \Omega$ we have

$$\Phi(\omega) = (s_1, s_2, s_3, \cdots)$$

where

$$s_0 = 1, \quad s_{n+1} = e_1, e_2, e_3, \cdots, e_n = s_n - 1, e_n.$$  \hfill (22)

For example,

$$\Phi(1, -1, 1, -1, 1, 1, 1, \cdots) = (1, -1, -1, 1, 1, 1, 1, \cdots).$$  \hfill (23)

The inverse mapping $\Phi^{-1}: \Omega \rightarrow \Omega$ is given by

$$\Phi^{-1}(s_1, s_2, s_3, \cdots) = (s_0, s_1, s_2, s_3, \cdots).$$  \hfill (24)

It is straightforward to prove that

$$\omega_1 < \omega_2 \text{ iff } \Phi(\omega_1) < \Phi(\omega_2).$$  \hfill (25)
2.3 Orthogonal Polynomials

One way of characterizing the invariant measure $\sigma$ when $2 \leq \lambda < \infty$ is by means of the associated set of monic orthogonal polynomials. We denote this set by $\{P_n(x)\}_{n=-1}^{\infty}$, where $P_{-1}(x) \equiv 0$. For $n \geq 0$, $P_n(x)$ has degree $n$ and the coefficient of $x^n$ is unity. The polynomials obey

$$\int_{\mathbb{R}} P_n(x) P_m(x) \, d\sigma(x) = 0 \quad \text{for} \quad n \neq m. \quad (1)$$

There are several reasons for interest in these polynomials. One is that, as we will see, they are related to the Tchebycheff polynomials $\{T_n(x) = \cos(n \cos^{-1} x)\}_{n=0}^{\infty}$ according to

$$\lim_{\lambda \to \frac{1}{2}^+} P_n(x) = 2 T_n\left(2x - 1\right) \quad (2)$$

Thus they provide an interesting generalization of the Tchebycheff polynomials. We note that the latter are the only family of $2F_1$ hypergeometric polynomials which are mapped into themselves under the change of variable $x \to (x-2)^2$.

More generally one would like a delineation of those families of monic orthogonal polynomials which are invariant under $x \to (x-\lambda)^2$.

The polynomials $\{P_n(x)\}_{n=-1}^{\infty}$ are also of interest because their measure is totally singular (it has no absolutely continuous part and no discrete jumps). To what extent does the singularity of their measure show up in their structure? In particular, is it possible via these polynomials to describe a Jacobi matrix (or any other self-adjoint operator) whose spectral density function is $\int_{\mathbb{R}} \sigma(dx)$? One would like finally to have on hand an explicit physical model associated
with \( \sigma \). We go a fair distance towards these objectives: we provide relations which allow all of the polynomials \( \{P_n(x)\}_{n=1}^{\infty} \) to be calculated, and we present certain unusual identities among the coefficients in the associated three term recurrence relation. We also display explicit formulas for \( P_{2n}(x) \) and \( P_{3,2n}(x) \), and note that the zeros of these polynomials can be written down. This may have a bearing on the problem of the determination both of the fixed points of \( T_m \) for positive integer values of \( m \), and also of the critical values of \( \lambda \) at which the real orbits under go bifurcation.

Let us introduce a second set of monic orthogonal polynomials \( \{Q_n(x)\}_{n=1}^{\infty} \) where \( Q_{-1}(x) \equiv 0 \). For \( n \geq 0 \), \( Q_n(x) \) has degree \( n \) and the coefficient of \( x^n \) is unity. The polynomials obey

\[
\int_{-\infty}^{\infty} Q_n(x)Q_m(x)\,dx = 0 \quad \text{for} \quad n \neq m. \tag{3}
\]

Then we will need the following result [8,12]. We include a proof since this is helpful towards understanding what follows.

**Lemma** For \( n \geq 0 \) and \( 2 \leq \lambda < \infty \)

\[
Q_n(x) = \frac{1}{x} \left[ P_{n+1}(x) - P_n(0) \frac{P_n(x)}{P_n(0)} \right]. \tag{4}
\]

*Proof:* First notice that the right-hand-side of (4) is always well-defined because \( P_n(0) \neq 0 \). This follows from the fact that the zeros of \( P_n(x) \) are confined to the interior of \( I \) which is the support of the measure. Now notice that the right-hand-side of (4) is a monic polynomial of degree \( n \). Finally observe that for \( n > m \geq 0 \)
because the polynomial of degree $m$ can be expressed as a linear combination of \{\{P_j(x) \mid j=0,1,\cdots,m\}\} all of which are orthogonal to both $P_{n+1}(x)$ and $P_n(x)$. Q.E.D.

We next define a set of polynomials $\{S_n(x)\}_{n=-1}^{\infty}$ by $S_{-1}(x) \equiv 0$ and

\begin{align*}
S_{2m}(x) &= P_m(x^2), \\
S_{2m+1}(x) &= x Q_m(x^2),
\end{align*}

for $m = 0,1,2,\cdots$ \hfill (6)

**Theorem 7** For $n \in \{-1,0,1,2,\cdots\}$

\[ S_n(x-\lambda) = P_n(x). \] \hfill (7)

**Proof.** Clearly $S_n(x-\lambda)$ is a monic polynomial of degree $n$, when $n \geq 0$.

It remains only to prove that $\{S_n(x-\lambda)\}_{n=-1}^{\infty}$ is a set of polynomials orthogonal with respect to $\sigma$. Consider first for $n \neq m$

\begin{align*}
\int_1 \frac{1}{x} \left[ P_{n+1}(x) - P_{n+1}(0) \frac{P_n(x)}{P_n(0)} \right] \frac{1}{x} \left[ P_{m+1}(x) - P_{m+1}(0) \frac{P_m(x)}{P_m(0)} \right] x \, d\sigma(x) \\
= \int_1 \left[ P_{n+1}(x) - P_{n+1}(0) \frac{P_n(x)}{P_n(0)} \right] \left[ P_{m+1}(x) - P_{m+1}(0) \frac{P_m(x)}{P_m(0)} \right] x \, d\sigma(x) \\
= 0
\end{align*}

because the polynomial of degree $m$ can be expressed as a linear combination of \{\{P_j(x) \mid j=0,1,\cdots,m\}\} all of which are orthogonal to both $P_{n+1}(x)$ and $P_n(x)$. Q.E.D.
where we have exploited the invariance of the measure $\sigma$ under $T_\lambda$ (Theorem 4). Next consider

$$\int I S_{2n}(x-\lambda) S_{2m}(x-\lambda) \, d\sigma(x)$$

$$= \int I P_n((x-\lambda)^2) P_m((x-\lambda)^2) \, d\sigma(x)$$

$$= \int I P_n(x) P_m(x) \, d\sigma(x) = 0$$

where again we have used Theorem 4. Finally consider, for $n$ and $m$ not necessarily distinct,

$$\int I S_{2n}(x-\lambda) S_{2m+1}(x-\lambda) \, d\sigma(x)$$

$$= \int I (x-\lambda) P_n((x-\lambda)^2) Q_m((x-\lambda)^2) \, d\sigma(x).$$

This is zero because the integrand is antisymmetric about the midpoint $\lambda$ of $I$ while $\sigma(x)$ is symmetric about $\lambda$.

Upon combining equations (4), (6) and (7) one finds

$$P_{2n}(x+\lambda) = P_n(x^2)$$

and

$$P_{2n+1}(x+\lambda) = \frac{1}{x} [P_{n+1}(x^2) - \frac{P_{n+1}^0}{P_n^0} P_n(x^2)].$$
These two equations provide a bootstrap procedure for calculating the \( P_n(x) \)'s, as indicated in the following scheme:

\[
1 = Q_0 \rightarrow S_1 \rightarrow P_1 \rightarrow P_2 \rightarrow P_4 \rightarrow P_8 \rightarrow \cdots \rightarrow P_{2n}(x) \rightarrow \cdots
\]

The numbers in parentheses indicate the equation to be used to travel along the arrows. Some examples of the resulting formulas are:

\[
P_1(x) = x - \lambda,
\]

\[
P_{2n}(x) = (\cdots((x-\lambda)^2-(\lambda)^2)\cdots-(\lambda)^2 - \lambda, \quad n \geq 1,
\]

\[
P_3(x) = (x-\lambda)((x-\lambda)^2-\lambda-1),
\]

\[
P_{3\cdot2n}(x) = P_{2n}(x)(P_{2m+1}(x)-1), \quad m \geq 0,
\]

\[
P_5(x) = (x-\lambda) \left[ (x-\lambda)^4 - \frac{(2x^2+2\lambda-1)}{\lambda-1} \right] (x-\lambda)^2 + \frac{\lambda^3-2\lambda^2+2\lambda+1}{\lambda-1}.
\]

Notice that the set of zeros of \( P_{2n}(x) \) is precisely the set \( K_n \) (section 2.1 equation (37)) which was used in the construction of \( \sigma \) (see below). Similarly, one can give expressions for all the zeros of \( P_{3\cdot2n}(x) \) and \( P_{5\cdot2n}(x) \).
We next consider the three term recurrence relation satisfied by the polynomials

THEOREM 8: For $2 < \lambda < \infty$ there exists a unique set of real numbers \( \{a_m\}_{m=1}^{\infty} \) such that for $n \in \{1, 2, 3, \ldots\}$

\[
P_{n+1}(x) = (x-\lambda) P_n(x) - a_n^2 P_{n-1}(x),
\]

(14)

\[
\lambda - a_{2n+1}^2 - a_{2n}^2 = 0,
\]

(15)

and

\[
a_n^2 = a_{2n}^2 a_{2n-1}^2.
\]

(16)

Proof. Since the measure $\sigma(x)$ is symmetric about $x=\lambda$ it follows at once from the theory of orthogonal polynomials that there exists a unique set of real numbers \( \{a_m\}_{m=1}^{\infty} \) such that the three term recurrence formula (14) holds. To prove (15) and (16) it is convenient to work in terms of the shifted polynomials

\[
S_n(x) = P_n(x+\lambda);
\]

we consider

\[
S_{2n+2}(x) = x S_{2n+1}(x) - a_{2n+1}^2 S_{2n}(x).
\]

(17)

Eliminating $S_{2n+1}(x)$ from this expression by using the recurrence relation we obtain

\[
S_{2n+2}(x) = (x^2 - a_{2n+1}^2) S_{2n}(x) - x a_{2n}^2 S_{2n-1}(x).
\]

(18)

Herein we reexpress $S_{2n-1}(x)$ in terms of $S_{2n}(x)$ and $S_{2n-2}(x)$, again with the aid of the recurrence equation, to obtain
\[ S_{2n+2}(x) = (x^2 - a_{2n+1}^2 - a_{2n}^2)S_{2n}(x) - a_{2n}^2 S_{2n-2}(x). \]  \hspace{1cm} (19)

We now set \( x' = x^2 - \lambda \) and use the fact that

\[ S_{2m}(x) = S_m(x^2 - \lambda) \] for \( m \in \{0, 1, 2, \ldots\}; \hspace{1cm} (20)\]

which yields

\[ S_{n+1}(x') = (x' + a_{2n+1}^2 - a_{2n}^2)S_n(x') - a_{2n}^2 S_{2n-1}(x'). \] \hspace{1cm} (21)

(15) and (16) follow at once upon comparing with the recurrence relation

\[ S_{n+1}(x) = xS_n(x) - a_n^2 S_{n-1}(x). \] \hspace{1cm} Q.E.D. \hspace{1cm} (22)

With the aid of (15) and (16) we readily calculate

\[ a_1^2 = \lambda, \quad a_2^2 = 1, \quad a_3^2 = \lambda - 1. \]

\[ a_4^2 = \frac{1}{\lambda - 1}, \quad a_5^2 = \frac{\lambda^2 - \lambda - 1}{\lambda - 1}, \quad a_6^2 = \frac{\lambda^2 - 2\lambda + 1}{\lambda^2 - \lambda - 1}. \] \hspace{1cm} (23)

We also obtain the following continued fractions representation for \( a_n^2 \): for \( n \geq 2 \)

\[ a_{2n}^2 = \frac{a_n^2}{1} - \frac{a_{n-1}^2}{\lambda} \ldots - \frac{a_2^2}{\lambda - 1}. \] \hspace{1cm} (24)

and
\[ a_{2n+1}^2 = \lambda - \frac{a_n^2}{|\lambda|} - \frac{a_{n-1}^2}{|\lambda|} - \cdots - \frac{a_2^2}{|\lambda|-1}. \]  

(25)

In particular we notice that when \( \lambda = 2 \) we have \( a_n^2 = 1 \) for \( n > 2 \) and \( a_1^2 = 2 \).

But then (14) is exactly the three term recursion relationship for the Tchebycheff polynomials \( \{T_n(\frac{1}{2}x-1)\}_{n=-1}^{\infty} \). This proves (2) and the first part of the next theorem.

**THEOREM 9:** For \( \lambda = 2 \), \( P_n(x) = 2 T_n(\frac{1}{2}x-1) \) for all \( n \in \{-1,0,1,2,\cdots\} \),

\( B_2 = [0,4] \), and the measure \( \sigma \) has for its cumulative distribution

\[ \sigma^{(2)}(x) = \begin{cases} 
0 & \text{for } x \leq 0, \\
\int_0^x \frac{dy}{\sqrt{y(4-y)}} & \text{for } 0 < x < 4, \\
1 & \text{for } x \geq 4.
\end{cases} \]  

(26)

**Completion of Proof:** The Tchebycheff polynomials \( 2 T_n(\frac{1}{2}x-1) \) are well known to be orthogonal monics associated with the distribution (26). We already know that the cumulative distribution function \( \sigma(x) \) provided by Lemma 5 is continuous, and when \( \lambda = 2 \) it vanishes for \( x < 0 \) and equals unity for \( x > 4 \). Since such a distribution must be unique we conclude \( \sigma^{(2)}(x) = \sigma(x) \) for all \( x \) when \( \lambda = 2 \). Finally since, by the Remark following Lemma 5, \( \frac{d\sigma(x)}{dx} \) exists and equals zero for all \( x \notin B_\lambda \) yet by (26) \( \frac{d\sigma^{(2)}(x)}{dx} \neq 0 \) for all \( x \in (0,4) \) and \( \frac{d\sigma^{(2)}(x)}{dx} \) does not exist for \( x = 0 \) and \( x = 4 \), we conclude that \( B_2 = [0,4] \). Q.E.D.
REMARK: Blumenthal's theorem [6] on the densification of the zeros of orthogonal polynomials upon the support of the measure is closely related to Theorem 9. From (13) it follows that the zeros of $T_{2n}(\frac{1}{2}x-1)$ are precisely the set of numbers

$$2 \pm \sqrt{(2 \pm \sqrt{(2 \pm \sqrt{...})})^{n \text{ times}}}.$$  

We are able to obtain further information about the family of approximating measures $\{\sigma_n(x)\}_{n=1}^{\infty}$ - which were constructed in Section 2.1, equation 40 - by examining the polynomials of the second kind, $\{p_n(x)\}_{n=0}^{\infty}$, which are linked with the measure $\sigma(x)$. These are defined by

$$p_n(x) = \int \frac{p_{n+1}(x) - p_{n+1}(y)}{x-y} \, d\sigma(y) \text{ for } n \in \{-1,0,1,2,\cdots\}. \quad (28)$$

**THEOREM 10** For all $n \in \{0,1,2,\cdots\}$,

$$p_{2n+1}^1(x) = (x-\lambda)p_n^1((x-\lambda)^2) \quad (29)$$

and

$$p_{2n}(x) = \frac{1}{(x-\lambda)} \left[ p_{2n+1}^1(x) - \frac{p_{2n+2}(\lambda)}{p_{2n}(\lambda)} \frac{p_{2n-1}^1(x)}{p_{2n-1}(x)} \right]. \quad (30)$$

**Proof.** From (28) one has

$$p_n^1((x-\lambda)^2) = \int \frac{P_{n+1}((x-\lambda)^2) - P_{n+1}((y-\lambda)^2)}{(x-\lambda)^2 - (y-\lambda)^2} \, d\sigma(y), \quad (31)$$
where the invariance of the measure under $T_\lambda$ has been exploited. We now
split up the denominator and use (11), which yields
\[
\frac{1}{n} \int \frac{P_{2n+2}(x) - P_{2n+2}(y)}{(x-\lambda) - (y-\lambda)} \, d\sigma(y)
\]
\[
+ \int \frac{P_{2n+2}(x) - P_{2n+2}(y)}{(x-\lambda) + (y-\lambda)} \, d\sigma(y).
\]
In the second integral here we make the change of variable $y-\lambda \to -(y-\lambda)$, use the symmetry of the measure and $I$ about $\lambda$, and again exploit (11), to provide
\[
\frac{1}{n} \int \frac{P_{2n+2}(x) - P_{2n+2}(y)}{(x-\lambda) - (y-\lambda)} \, d\sigma(y) \quad \text{(32)}
\]
From this (29) is immediate.

Equation (28) also implies the recursion relation
\[
P_{n+1}^1(x) = (x-\lambda)P_n^1(x) - a_{n+1}^2 P_{n-1}^1(x) \quad \text{for } n \in \{0,1,2,\ldots\},
\]
\[
\text{with } P_0^1(x) = 0, \text{ and } P_1^1(x) = 1. \text{ This implies (30) when } a_{n+1}^2 \text{ is eliminated with the aid of (14) wherein } \lambda = x. \quad \text{Q.E.D.}
\]

From (29) it is apparent, in contrast to the previous case,
that the odd polynomials of the second kind are easily calculated from the even ones. Some examples of the polynomials are

\text{Example of odd polynomials of the second kind}:
Now consider the moment functions

\[ G(x) = \int \frac{d\sigma(x)}{x-y}. \]  

and

\[ G(x) = \int \frac{d\nu(x)}{x-y}. \]  

From the theory of Padé approximants [2], one has that the \([n-1/n](x)\) approximant to \(G(x)\) is

\[ [n-1/n](x) = \frac{p_{n-1}(x)}{p_n(x)}, \quad \text{for } n \in \{1,2,\cdots\}. \]  

Using (11) and (20) we discover the remarkable result
\[ [2n-1/2n](x) = (x-\lambda)[n-1/n][(x-\lambda)^2], \text{ for } n \in \{1, 2, \cdots\}. \quad (39) \]

Also from (38) and (35) with \( n = 2^K, K \in \{0, 1, 2, \cdots\} \), we find

\[ [2^K-1/2^K](x) = \frac{1}{2^K} \frac{d}{dx} \ln P_{2^K}(x) = \int_1^x \frac{d\sigma_n(x)}{x-y}. \quad (40) \]

Finally, we note that since \((B_\lambda, \overline{B}_\lambda, \sigma_T)\) is a mixing system, so is \((B_\lambda, \overline{B}_\lambda, \sigma_T^n)\) for \( n \in \{1, 2, 3, \cdots\} \). Hence \( P_{2^n}(x+\lambda) \) provides a mixing transformation on \( B_\lambda \), with respect to \( \mu \). Shifting \( B_\lambda \) to the right by subtracting \( \lambda \), and correspondingly adjusting the measure, this shows that each of the polynomials \( P_{2^n}(x+\lambda) \) provides a mixing transformation upon the shifted system. This extends a result of Alder and Rivlin [1], and is itself a special case of a wide reaching theorem [4].
The cases $-\frac{1}{4} \leq \lambda \leq 2$, and $|\lambda| \leq \frac{1}{4}$ with $\lambda \in \mathbb{E}$

3.1 Two constructions for $B_{\lambda}$.

We present two constructions for $B_{\lambda}$; one from the "outside", and one from the "inside", when $-\frac{1}{4} \leq \lambda \leq 2$. The first is not new in principle: it involves the formation of an increasing sequence of domains, successive inverse images under $T_{\lambda}$ of a neighborhood of $\infty$, as suggested by Fatou [10] and by Julia [16]. What is novel, and somewhat special to $T_{\lambda}$, is the possibility of carrying out this construction explicitly. We make a sequence of analytic functions, mapping the exterior of the unit circle $D_0$ to the sequence of domains, which converges to an analytic function $F_{\lambda}(z)$. The latter takes $D_0$ conformally onto a domain $D_{\lambda}$, and obeys (1.12). The boundary of $D_{\lambda}$ is $B_{\lambda}$. Our procedure shows constructively how the action of $T_{\lambda}$ on $B_{\lambda}$ is connected to that of $T_0$ on $B_0$ (the unit circle). This, not surprisingly, turns out to be fundamental to the establishment of general isomorphism properties, when $\lambda < 2$ for example, see [4]. The relationship between $B_{\lambda}$ and $B_{\lambda}$ (once a branch cut has been fixed), is implied.

The second construction, from the "inside", provides a decreasing sequence of domains and a corresponding sequence of functions, from $D_0$ to the domains, converging uniformly to $F_{\lambda}(z)$ on compact subsets of $D_0$. Of interest are the complements of the domains, which form an increasing sequence of trees, with fractal-like structure [18] and two-dimensional measure zero, which serve to describe $B_{\lambda}$ from the interior. This construction turns out to be important: we have recently proved [4] that this sequence of trees converges to $B_{\lambda}$ itself, for infinitely many values of $\lambda \in (0,2)$.

A preliminary example illustrates how it is that one cannot find an analytic mapping of $B_0$ onto $B_{\lambda}$ in general, and why one must be satisfied with approaches from the interior and exterior. Recall that $B_0$ is the unit circle, $B_0 = \{z \in \mathbb{E} | |z| = 1\}$, while $B_2$ in the real interval $[0, 4]$. The mapping
\( F_2(z) = z + z^{-1} + 2 \quad (1) \)

takes \( B_0 \) into \( B_2 \), but as \( z \) goes once around the circle, \( F_2(z) \) traverses [0,4] twice. Notice however that \( F_2(z) \) is a one-to-one analytic mapping from the exterior of the unit circle to the exterior of the interval [0,4].

We will use the notation \( \hat{\mathbb{C}} \) for the extended complex plane, \( \hat{\mathbb{C}} = \mathbb{C} \cup \{0\} \), and we denote

\[
D_0 = \{ z \in \hat{\mathbb{C}} \mid |z| > 1 \} \quad (2)
\]

For \( \lambda \geq -\frac{1}{4} \) we define...
a = \sqrt{\lambda + \frac{1}{4}} + \lambda + \frac{1}{2}, \text{ and } b = \sqrt{|\lambda| + \frac{1}{4}} + |\lambda| + \frac{1}{2}. \tag{3}

Then \( a \) is the unique positive real numbers which obeys \( a = \lambda + \sqrt{a} \). Notice that \( a \leq b \), and \( \lambda - \sqrt{a} \leq 0 \), for \( -\frac{1}{4} \leq \lambda \leq 2 \).

**Lemma 8.** Let \(-\frac{1}{4} \leq \lambda \leq 2 \) and \( n \in \{0,1,2,\cdots\} \). Then the following five part statement, \( S_n \), is true. \( S_n(i) \): There exists a well defined one-to-one analytic function \( f_n : D_0 \rightarrow \mathbb{C} \) such that

\[
  f_n(z) = b^2 (\frac{1}{z})
\]

\[
  f_n(z) = \begin{cases} 
    \lambda + \sqrt{b} z & \text{if } n = 0, \\
    \lambda + \sqrt{f_{n-1}(z)} & \text{if } n > 0.
  \end{cases}
\]

\( S_n(ii) \): \( f_n(z) - \lambda \) is an odd function of \( z \).

\( S_n(iii) \): \( f_n(\{x \mid x > 1\}) \subset \{x \mid x > a\} \).

\( S_n(iv) \): \( \{x \mid \lambda - \sqrt{a} \leq x \leq a\} \cap f_n(D_0) = \emptyset \)

\( S_n(v) \): \( f_n(D_0) \supset f_{n-1}(D_0) \) for \( n > 0 \).

**Proof.** The proof is by induction. For \( n = 0 \) the function \( f_0(z) = \lambda + \sqrt{b} z \) is provided explicitly by (5), and it obeys (4); so \( S_0(i) \) is true. \( S_0(iii) \) and \( S_0(iv) \) are readily demonstrated with the aid of the inequality \( a \leq b \).

The remaining parts of \( S_0 \) are true by inspection.

We now assume that \( S_n \) is true for all \( n \in \{0,1,\cdots,N-1\} \) for some positive integer \( N \). Then \( G : D_0 \rightarrow \mathbb{C} \) defined by
\[ G(z) = b^{-2N} f_{N-1}(z) = z + b^{-2N} \lambda + O(1/z) \] (6)

is one-to-one and analytic. Moreover by \( S_{N-1}(iv) \) we have \( 0 \not\in G(D_0) \) whence \( g: \{ z \mid |z| < 1 \} \to \mathbb{C} \) defined by

\[ g(z) = 1/G(1/z) \] (7)

is one-to-one, analytic, and has the property \( g(z) = z + O(z^2) \). It now follows from Goluzin [13, P. 48] that there exists a one-to-one analytic function \( h:\{ z \mid |z| < 1 \} \to \mathbb{C} \) which is a branch of \( \sqrt{g(z^2)} \) and which obeys \( h(z) = z + O(z^3) \). Next we define \( H:D_0 \to \mathbb{C} \) by

\[ H(z) = 1/h(1/z). \] (8)

Then it follows that \( H(z) \) is a one-to-one analytic function which is a branch of \( \sqrt{G(z^2)} \) and such that \( H(z) = z + O(1/z) \). This shows that \( S_N(i) \) is satisfied by defining \( f_N:D_0 \to \mathbb{C} \) by

\[ f_N(z) = b^{2(N+1)} H(z) + \lambda. \] (9)

To prove \( S_N(ii) \) we observe that \( f_{N-1}(z) \) has a power series at \( \infty \) of the form

\[ f_{N-1}(z) = b^{-2N} \left( z + \sum_{j=0}^{\infty} c_j z^{-2j} \right) \] (10)
from which it follows that

\[ f_N(z) = \lambda + b \cdot 2^{-(N+1)} \cdot z \sqrt{1 + \sum_{j=0}^{\infty} c_j z^{-2(j+1)}} \]  

This shows that \( f_N(z) - \lambda \) is an odd function as desired.

From the normalization at \( -\infty \) in \( S_N(i) \) the square root in (11) must be positive when its contents is positive. Hence for \( x > 1 \) we have, from \( S_{N-1}(iii) \),

\[ f_N(x) = \lambda + \sqrt{f_{N-1}(x)} > \lambda + \sqrt{\frac{1}{a}} = a \]  

which implies \( S_N(iii) \) is true.

Since \( f_N(z) - \lambda \) is odd it now follows that

\[ f_N(\{ x | x < -1 \}) \subset \{ x | x < \lambda - \sqrt{a} \}. \]  

Thus if \( z \) is real and \( z \in D_0 \) then \( f_N(z) \notin [\lambda - \sqrt{a}, a] \). The proof of \( S_N(iv) \) is completed if we show that whenever \( z \in D_0 \) and \( z \neq \bar{z} \) then also \( f_N(z) \neq f_N(\bar{z}) = \overline{f_N(z)} \) where we use the reality of \( f_N(z) \) on the real line.

But \( f_N(z) \) is one-to-one so this is immediate.

We now prove \( S_1(v) \). For all \( z \in \partial D_0 = \{ z \in \mathbb{C} | |z| = 1 \} \) we have

\[ |f_1(z) - \lambda| = |\sqrt{f_0(\bar{z})}| = |\sqrt{\lambda + \sqrt{b}z^2}| \leq |\sqrt{\lambda} + \sqrt{b}| = \sqrt{b}, \]  

whilst for all \( z \in D_0 \) we have
\[
|f_0(z) - \lambda| = \sqrt{6} \quad |z| > \sqrt{6} .
\] (15)

Hence \( f_1(\partial D_0) \cap f_0(D_0) = \emptyset \). Since \( f_1(D_0) \) is a simply connected domain containing \( \infty \) and bounded by \( f_1(\partial D_0) \) we deduce \( f_1(D_0) \supset f_0(D_0) \).

In order to prove \( S_N(v) \) for \( N > 1 \) we assume \( S_{N-1}(v) \) is true so that

\[
f_{N-1}(D_0) \supset f_{N-2}(D_0) .
\] (16)

Let \( z \in D_0 \). Then \( z^2 \in D_0 \) and by (16) there is a \( \tilde{w} \in D_0 \) such that \( f_{N-1}(\tilde{w}) = f_{N-2}(z^2) \). But there exists \( w \in D_0 \) such that \( w^2 = \tilde{w} \) because \( D_0 = T_0^{-1}D_0 \) and hence

\[
f_{N-1}(w^2) = f_{N-2}(z^2) \quad \text{with} \quad w \in D_0 .
\] (17)

This implies

\[
f_{N-1}(z) - \lambda = \sqrt{f_{N-2}(z^2)} = \sqrt{f_{N-1}(w^2)} = \pm (f_N(w) - \lambda) ,
\] (18)

where the \( \pm \) sign indicates the possible choice of branches. But since \( f_N(w) - \lambda \) is an odd function of \( w \), by \( S_N(\text{ii}) \) it follows that

\[
\pm (f_N(w) - \lambda) = f_N(\pm w) - \lambda .
\] (19)

Hence \( f_{N-1}(z) \) equals either \( f_N(w) \) or \( f_N(-w) \). Since both \( w \) and \( -w \) belong to \( D_0 \), it now follows that \( f_{N-1}(z) \in f_N(D_0) \) which proves \( S_N(v) \). Q.E.D.
THEOREM 11 Let $-1 \leq \lambda \leq 2$. Then there exists a function $F_{\lambda} : D_0 \to \hat{\mathbb{C}}$ such that the following statements are true.

(i) The sequence of functions $\{f_n\}_{n=0}^{\infty}$ provided by Lemma 8 converges uniformly to $F_{\lambda}$ on compact subsets of $D_0$. $F_{\lambda}$ maps $D_0$ conformally onto some domain $D_{\lambda}$.

(ii) There exists a function $G \in L^2[-\pi, \pi]$ such that

$$\lim_{r \to 1+} F_{\lambda}(re^{i\theta}) = G(\theta) \quad \text{a.e.} \quad (20)$$

(iii) Let $I = \{e^{i\theta} \mid \lim_{r \to 1+} F_{\lambda}(re^{i\theta}) = G(\theta)\}$.

Then

$$F_{\lambda}(z) = \lambda + \sqrt{P_{\lambda}(z^2)} \quad \text{for all } z \in D_0 \cup \Gamma, \quad (21)$$

and

$$T_{-1}^{-1} B_{\lambda} = B_{\lambda} \quad (22)$$

where $B_{\lambda}$ denotes the boundary of $D_{\lambda}$.

Proof. From $S_n$ (iv) and $S_n$ (v) we find that the image sets $f_n(D_0)$ are increasing and that their union is disjoint from the interval $[\lambda - \sqrt{\lambda}, \lambda]$. (1) now follows from Carathéodory's Theorem on Domain Convergence, in Goluzin [13, P. 55], applied to the sequence $\{1/f_n(1/z)\}$. 
To establish properties of $F_\lambda$, we can let $n \to \infty$ in Lemma 8. For example, from the first part of $S_n(i)$ we can determine the leading terms in the Laurent series for $F_\lambda(z)$:

$$F_\lambda(z) = z + \lambda + \sum_{j=1}^{\infty} a_j/z^j. \quad (23)$$

The area theorem given by Hille [14, P. 347] permits us to conclude

$$\sum_{j=1}^{\infty} j|a_j|^2 < \infty \quad (24)$$

Let

$$f(z) = F_\lambda(1/z) - 1/z = \lambda + \sum_{j=1}^{\infty} a_j z^j. \quad (25)$$

Then $f$ is analytic for $\{z \mid |z| < 1\}$, and $\sum_{j=1}^{\infty} |a_j|^2 < \sum_{j=1}^{\infty} j|a_j|^2 < \infty$. Hence, from Hille [14, P. 364] it follows that

$$1 + \sum_{j=1}^{\infty} a_j e^{ij\theta} \quad (26)$$

is the Fourier series of a function $g \in L^2[-\pi, \pi]$ and that

$$\lim_{r \to 1^-} f(re^{i\theta}) = g(\theta) \text{ a.e.} \quad (27)$$

We now define

$$G(\theta) = g(-\theta) + e^{i\theta}. \quad (28)$$
Then using the facts that $F^\lambda(z) = f(1/z) + z$ and $\lim_{r\to 1^-} r^{-i\theta}$, we readily verify that (ii) is true.

In the limit as $n \to \infty$, $S_n(i)$ implies (21) for $z \in D_0$. By taking limits as $r \to 1$ using (20) we find (21) also holds for $z \in T$.

From $S_n(ii)$ we deduce that $F^\lambda(z) - \lambda$ is an odd function of $z$. This, together with (21), implies that for $z \in D_0$

$$T^{-1} F^\lambda(z) = \{ \lambda + \sqrt{F^\lambda(z)} , \lambda - \sqrt{F^\lambda(z)} \} = \{ F^\lambda(\sqrt{z}) , F^\lambda(-\sqrt{z}) \}.$$ (29)

Since $z \in D_0$ if and only if both $\sqrt{z}$ and $-\sqrt{z}$ belong to $D_0$, we have $T^{-1} D_{\lambda} = D_{\lambda}$. Because both branches of $T^{-1}$ are continuous, this implies $T^{-1} D_{\lambda} = \tilde{D}_{\lambda}$. The latter two equations imply (22), Q.E.D.

By combining the normalization condition

$$F^\lambda(z) = z + \lambda + O(1/z),$$ (30)

and the functional equation

$$F^\lambda(z) = \lambda + \sqrt{F^\lambda(z^2)},$$ (31)

We can calculate in recursive fashion the power series expansion of $F^\lambda(z)$ about the point at infinity. Thus
This iterative procedure can be continued indefinitely. The nth iteration produces $2^{n-1}$ further series coefficients. The finally obtained series will of course be convergent for all $z \in D_0$. Notice that the construction here shows in particular that there is only one analytic function $F_{\lambda} : D_0 \to \hat{C}$ which obeys (30) and (31).

When $\lambda = 2$ we have $F_2(z) = z + 2 + \frac{1}{z}$ which shows that $B_2 = [0,4]$ just as it should. This assures us that the set $B_{\lambda}$ described for $\lambda \geq 2$ is indeed the continuation of the one obtained in this section.

Note that the coefficient of $1/z^5$ in (33) is strictly negative for $0 < \lambda < 2$. Since $F_\lambda(1) = a$, and $a$ is an extreme point of the set $B_{\lambda}$, there is some hope that for all sufficiently large $n$ (at fixed $\lambda$) the coefficient of $1/z^{2n+1}$ is non-negative. Further analysis of the induction formulas may establish this. Such a result, combined with the fact that $\lim_{x \to 1^+} F_{\lambda}(x)$ exists, would allow us to conclude that the series for $F_{\lambda}$ is absolutely convergent on the unit circle and thus that the boundary values are defined everywhere rather than almost everywhere.

In order to obtain more detailed information about $B_{\lambda}$ it is useful to construct a second sequence of functions $\{f_n(z)\}_{n=0}^{\infty}$ which is convergent to $F_{\lambda}(z)$. Whereas the sequence $\{f_n(z)\}_{n=0}^{\infty}$ provides a sequence of images which
increases to $D^\lambda$ (whose boundary is $B^\lambda$), the sequence $\left\{ f_n^*(z) \right\}_{n=0}^\infty$ yields decreasing images, and we get convergence to $B^\lambda$ from the "inside".

**Lemma 9** Let $\frac{1}{2} \leq \lambda \leq 2$ and $n \in \{0, 1, 2, \ldots\}$. Then the following five part statement, $S_n^\ast$, is true. $S_n^\ast(i)$: There exists a well defined one-to-one analytic function $f_n^* : D^0 \to \widehat{C}$ such that

$$f_n^*(z) = (a/4)^{-(n+1)} z + \lambda + O(1/z)$$

and

$$f_n^*(z) = \begin{cases} 
\lambda + (a/4) (z + 1/z) & \text{if } n=0 \\
\lambda + \sqrt{f_{n-1}^*(z^2)} & \text{if } n > 0.
\end{cases}$$

$S_n^\ast(ii)$: $f_n^*(z) - \lambda$ is an odd function of $z$.

$S_n^\ast(iii)$: $f_n^*([x]x > 1)) \subset \{x|x > a\}$.

$S_n^\ast(iv)$: $\{x|\lambda - \sqrt{a} \leq x \leq a\} \cap f_n^*(D^0) = \emptyset$.

$S_n^\ast(v)$: $f_n^*(D^0) \subset f_{n-1}^*(D^0)$ for $n > 0$.

**Proof.** The proof parallels that of Lemma 8, so we will mention only the differences.
The explicit function \( f_0^*(z) \) maps \( D_0 \) onto the complement of \([\lambda-\sqrt{a}, a]\), and one readily verifies that all of the statements are true for \( n=0 \). The induction step which yields \( S^*_n(i)-(iv) \) is essentially the same as that for \( S^*_n(i)-(iv) \). The main and final point to check is the set inclusion in \( S^*_n(v) \) which is opposite to the one in \( S^*_n(v) \).

Let \( \Gamma_n = f_n^*(\partial D_0) \) denote the complement of \( f_n^*(D_0) \). Then \( S^*_n(v) \) can be restated as the claim that the sequence of sets \( \{\Gamma_n\}_{n=0}^{\infty} \) is increasing, according to

\[
\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots
\]

(36)

It is straightforward to show that \( \Gamma_0 \subset \Gamma_1 \) (see below). One next assumes

\[
\Gamma_{N-1} = f_{N-1}^*(\partial D_0) \supset f_{N-2}^*(\partial D_0) = \Gamma_{N-2}
\]

for some integer \( N > 1 \) and continues the induction exactly as in (17) and (18), but with \( D_0 \) replaced by \( \partial D_0 \) and \( f_n \) replaced by \( f_n^* \).

Q.E.D.

It turns out to be very useful to give a more complete characterization of the sets \( \Gamma_n \) than the one above. Let \( S_0 = [\lambda-\sqrt{a}, a] = \Gamma_0 \). Then the image of \( S_0 \) under the principal branch of \( \lambda + \sqrt{z} \) is \([\lambda, a] \cup \text{the closed line segment from } \lambda \text{ to } \lambda + 1\sqrt{a} - \lambda \). Since \( f_1^*(z) - \lambda \) is an odd function of \( z \) we can reflect the latter set about \( \lambda \) to obtain the rest of \( \Gamma_1 \). In this way we find

\[
\Gamma_1 = S_0 \cup S_{11}
\]

(37)

where \( S_{11} \) is the closed line segment connecting \( \lambda \pm \sqrt{a} - \lambda \). It will be found helpful here to look at Figure 1. \( \Gamma_2 \) is the image of \( \Gamma_1 \) under \( \lambda \pm \sqrt{z} \). \( S_0 \)
will again generate $S_0 \cup S_{11}$ but $S_{11}$ will produce two new analytic arcs denoted by $S_{21}$ and $S_{22}$. In general $\Gamma_n$ will contain $2^{n-1}$ new arcs not in $\Gamma_{n-1}$:

$$\Gamma_n = S_0 \cup \bigcup_{j=1}^{2^{j-1}} (\bigcup_{k=1}^{2^{j-1}} S_{j,k})$$

(38)

where each $S_{j,k}$ is an analytic arc. The endpoints of $S_0$ are $\lambda \pm \sqrt{\alpha} = F_\lambda(\pm 1)$, and by repeatedly using the fact that $F_\lambda(z) = \lambda + \sqrt{F(z^2)}$ together with

$$f_n^*(z) = \lambda + \sqrt{f_{n-1}^*(z^2)}$$

we find that the endpoints of the arcs composing $\Gamma_n$ are $F_\lambda(e^{j\pi i/2^n})$ for all $j \in \{1,2,\ldots,2^{n+1}\}$. Equivalently, the endpoints of $\Gamma_n$ are all numbers expressible in the form

$$F((\pm \sqrt{\cdots \sqrt{\pm 1}} \cdots )) = \lambda \pm \sqrt{\lambda \pm \sqrt{\cdots \sqrt{\lambda \pm \sqrt{\alpha}} \cdots}}$$

(39)

where $\pm$ indicates that both branches of $\sqrt{\cdots}$ can be chosen.

We may now parallel the proof of the first part of Theorem 11 to obtain the following result.

**THEOREM 12** Let $-1 \leq \lambda \leq 2$. Then the sequence of functions $\{f_n^*\}_{n=0}^\infty$ provided by Lemma 9 converges uniformly to $F_\lambda$ on compact subsets of $D_0$.

**Proof.** Carathéodory's Theorem on Domain Convergence again applies, showing that $\{f_n^*\}_{n=0}^\infty$ converges uniformly on compact subsets of $D_0$ to some analytic function $F_\lambda^*$. But $F_\lambda^*$ satisfies exactly the same functional equation and normalization condition as are satisfied by $F_\lambda$, whence $F_\lambda = F_\lambda^*$. Q.E.D.
If $K$ is a compact subset of $D_0$, then the sequence of sets $\{f_n(K)\}_{n=0}^{\infty}$ and $\{f_n^*(K)\}_{n=0}^{\infty}$ increases and decrease respectively to $F_\lambda(K)$. Similarly $\{f_n(D_0)\}_{n=0}^{\infty}$ increases to $F_\lambda(D_0) = D_\lambda$. However, the decreasing sequence $\{f_n^*(D_0)\}_{n=0}^{\infty}$ does not converge to $D_\lambda$. This sequence illustrates a peculiar property of domain convergence. For example, we will show in the next section that when $0 \leq \lambda < 1$, $D_\lambda^C$ contains the disk of radius $\frac{1}{2}$ about $0$. In this case the area of $D_\lambda^C$ is strictly positive, yet the area of $\Gamma_n$ is zero for all $n$.

The behavior of these image sets can be better understood by considering the endpoints of the analytic arcs in $\Gamma_n$. The set $\left\{ \bigcup_{n=1}^{\infty} \frac{e^{j\pi/2n}}{j = 1, 2, \ldots, 2^{n+1}} \right\}$ is dense in the unit circle $\partial D_0 = B_0$, and its image under $F_\lambda$ is dense in $B_\lambda$. The latter image is precisely the set of endpoints of all of the analytic arcs in all of the $\Gamma_n$'s. Thus, consistently with the general theory of domain convergence, we find $F_\lambda(D_0)$ is one component of the interior of $\lim_{n \to \infty} f_n^*(D_0)$.

For $0 < \lambda < 2$ the numbers $\lambda \pm \sqrt{a}$ are on the boundary of both $f_0(D_0)$ and $f_0^*(D_0)$. Hence

$$f_n(z) = f_n^*(z) = F_\lambda(z) \quad \text{for} \quad z = e^{j\pi/2n}$$

(40)

for all $j \in \{1, 2, \ldots, 2^{n+1}\}$ and $n \in \{0, 1, 2, \ldots\}$. If we superimpose the boundaries of $f_n(D_0)$ and $f_n^*(D_0)$ then we separate the complex plane into many components. The portion of $B_\lambda$ corresponding to $\{e^{j\theta}|(j-1)\pi/2^n \leq \theta \leq j\pi/2^n\}$ must lies in the closure of the jth component counted counter-clockwise, starting from the one in the first quadrant which has a on its boundary. As an example we illustrate the case $n=3$ in Figure 2 where the sixteen components are labeled.
3.2 The relation between \( B_\lambda \) and \( \hat{B}_\lambda \) when \(|\lambda| < \frac{1}{4}\) with \( \lambda \in \mathbb{C} \).

Here, for \(|\lambda| < \frac{1}{4}\) with \( \lambda \in \mathbb{C} \), we construct \( B_\lambda \), neither from the "outside" nor from the "inside", by means of convergent sequences of approximants for the individual elements of the set. Thus, \( B_\lambda \) is connected to \( \hat{B}_\lambda \) in much the same way as it was in §2.1. There are important differences however. In §2.1 the approximants converged to particles of a Fatou dust, whilst here they will approach elements of a Holder continuous curve. Also, the mapping from \( B_\lambda \) to \( \hat{B}_\lambda \) is no longer one-to-one. Part of what we do in this section is to realize the construction mentioned by Fatou [10, §24] over a range of \( \lambda \)-values.

First, we restrict attention to the case \( 0 \leq \lambda \leq \frac{1}{2} \); in this situation \( F_\lambda(z) \) is defined for all \( z \in \mathbb{C} \) such that \(|z| = 1\), and we show how the value of \( F_\lambda(e^{i\theta}) \) is identified with a member of \( \hat{B}_\lambda \). Second, we extend consideration to the case \( \lambda \in \mathbb{L} \),

\[
L = \{ \lambda \in \mathbb{C} \mid |\lambda| < \frac{1}{4} \},
\]

by showing that \( F_\lambda(z) \) is both defined and analytic in \( \lambda \), for \( \lambda \in \mathbb{L} \). \( B_\lambda \) is thus defined for \( \lambda \in \mathbb{L} \). Last, we show that the family of functions

\[
G_\lambda(\theta) = F_\lambda(e^{i\theta}) \text{ where } \lambda \text{ belongs to a compact subset of } \mathbb{L},
\]

is uniformly Hölder continuous in \( \theta \).

Let \( S_0(z) = z \), and for \( n \in \{1, 2, 3, \ldots\} \) define a function of \( z \in \mathbb{C} \) by

\[
S_n(z) = \lambda + e_1 \sqrt{(\lambda + e_2 \sqrt{(\lambda + \cdots + e_{n-1} \sqrt{(\lambda + e_n \sqrt{\lambda})})})}
\]

(2)

The value of the square root \( \sqrt{w} \) for \( w \in \mathbb{C} \) is fixed by writing \( w = re^{i\theta} \) with \( 0 \leq \theta \leq 2\pi \), and then

\[
\sqrt{z} = \sqrt{r} e^{i\theta/2} \quad \text{and} \quad -\sqrt{z} = -\sqrt{r} e^{i\theta/2} = \sqrt{r} e^{i(\theta+2\pi)/2}. \]

(3)

We will say that we have a positive axis branch cut.

Footnote* Here we use the nomenclature of Mandelbrot [18].
$S(n, z)$ is clearly a well defined function of its argument. In order to emphasize its dependence upon the sequence $\omega \in \Omega$ we will write

$$S(n, z) = S_n(\omega, z)$$

(4)

which is similar to the usage in §2.1. Also, as there, we will use the notation

$$\mathcal{S}_n = \{S_n(\omega, z) | \omega \in \Omega\}$$

(5)

We introduce the special notation

$$S(+, z) = \lambda + \sqrt{z} \text{ and } S(-, z) = \lambda - \sqrt{z}.$$  

(6)

Finally, for $-1 < \lambda < 2$, we define

$$S(\omega) = \lim_{n \to \infty} S_n(\omega, a)$$

(7)

when the limit exists, where $a$ is defined in Section 3.1 Equation (6). Note that $a$ is the fixed point of $S(+, z)$.

**Lemma 10** Let $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$, with $z \neq 0$. Then

$$S(+, z) = S(-, \bar{z})$$

**Proof.** Let $z = r e^{i\theta}$ with $0 < \theta \leq 2\pi$. Then
LEMMA 11 Let $\lambda > 0$ and $d > a$. Then both $S(\lambda, z)$ and $S(-\lambda, z)$ map the set \( \{z \mid |z| \leq d\} \) into itself.

Proof. Since for $d > a$ we have $(\lambda - d)^2 \geq d$, it follows that $S(\lambda, d) \leq d$. Hence for $|z| \leq d$ we have

\[
|S(\pm, z)| = |\lambda \pm \sqrt{z}| \leq \lambda + \sqrt{|z|} \leq \lambda + \sqrt{d} \leq d
\]

Q.E.D.

LEMMA 12 Let $\lambda \in L$ and let

\[
c = -|\lambda| + \frac{1}{2} + \sqrt{-|\lambda|} + \frac{1}{2}
\]

Then both $S(\lambda, z)$ and $S(-\lambda, z)$ map the set \( \{z \mid |z| \geq c\} \) into itself.

Proof. Note that $c > \frac{1}{2} - |\lambda| > \frac{1}{2} > |\lambda|$, and let $|z| \geq c$. Then

\[
|S(\pm, z)| \geq |\sqrt{z} - |\lambda|| = \sqrt{|z|} - |\lambda|
\]

\[
\geq \sqrt{c} - |\lambda| = c.
\]

Q.E.D.
Let \( \lambda \in \mathbb{L} \) and \( d \geq a \), and set
\[
R_d = \{ z | c \leq |z| \leq d \} \quad (13)
\]
Then Lemmas 11 and 12 together imply that both \( S(\pm z) \) and \( S(-,z) \) map \( R_d \) into itself. Let \( \mu \), \( \sigma \), and \( \xi \) in \( \Omega \) be defined by
\[
\begin{align*}
\mu &= (-1,-1,-1,-1,\ldots), \\
\sigma &= (+1,-1,-1,-1,\ldots), \\
\xi &= (-1,+1,+1,+1,\ldots).
\end{align*}
\]
and \( \xi = (-1,+1,+1,+1,\ldots) \). \quad (14)

**Lemma 13** For \( 0 < \lambda \leq 2 \), \( S(0) \) and \( S(\xi) \) both exist and \( S(0) = S(\xi) = S(-,a) \).

**Proof.** The existence of \( S(\xi) \) and the second equality are immediate because \( S(+,a) = a \) so that for all \( n \in \{2,3,4,\ldots\} \)
\[
S_n(\xi,a) = S_{n-1}(\xi,S(+,a)) = S_{n-1}(\xi,a). \quad (15)
\]
We now claim that \( S(-,z) \) is a contraction mapping towards \( a \) on \( \{z | \text{Im}z < 0\} \)
and that \( S(+,z) \) is a contraction mapping towards \( a \) on \( \{z | \text{Im}z > 0\} \). By Lemma 10 these claims are equivalent, so we only prove the latter. But for \( \text{Im}z > 0 \) we have
where we have used the fact that \( \text{Re} \sqrt{z} > 0 \). This proves our claim. It now follows that the sequence \( \{ \mathcal{S}_n(\mu, a) \}_{n=1}^{\infty} \) approaches \( \lambda + \sqrt{a} \) through the fourth quadrant, which, in turn implies that \( \{ \sqrt{\mathcal{S}_n(\mu, a)} \}_{n=1}^{\infty} \) converges to \( -\sqrt{a} \). Hence \( \{ \mathcal{S}_n(\sigma, a) \}_{n=1}^{\infty} \) converges to \( \lambda - \sqrt{a} \).

Q.E.D.

**Lemma 14** Let \( 0 < \lambda < 0.2, d > a, n \in \{1, 2, \cdots \}, \) and \( \mathcal{S}_n(\omega, z) \in \mathcal{B}_n \). Then for any \( z_1, z_2 \in \mathbb{R}^d \),

\[
|\mathcal{S}_n(\omega, z_1) - \mathcal{S}_n(\omega, z_2)| \leq 2dr^{n-3}
\]  

(17)

where \( r = 0.87 \).

**Proof.** First note that for any \( \omega_1, \omega_2 \in \mathbb{R}^d \) we have

\[
|\arg \sqrt{\omega_1} - \arg \sqrt{\omega_2}| = |\arg(-\sqrt{\omega_1}) - \arg(-\sqrt{\omega_2})| = \frac{1}{2} |\arg \omega_1 - \arg \omega_2|
\]  

(18)

Also, for any \( w \in \mathbb{R}^d, \sqrt{w} \) lies in the upper half plane and
0 \leq \arg \sqrt{w} - \arg(\sqrt{w} + \lambda) \\
\leq \sin^{-1}\left(\frac{\lambda}{\sqrt{|w|}}\right) \leq \sin^{-1}\left(\frac{0.2}{\sqrt{0.5}}\right) < 0.3 \tag{19}

Here we have used the fact that when 0 < \lambda < 0.2, then c \geq 0.5 so that for all w \in \mathbb{R} we have |w| > 0.5. We now obtain

|\arg S(\pm, w_1) - \arg S(\pm, w_2)| = |\arg(\lambda + \sqrt{w_1}) - \arg(\lambda + \sqrt{w_2})| \\
\leq \frac{1}{2} |\arg w_1 - \arg w_2| + 0.3. \tag{20}

Similar reasoning applies to S(-, w), and thus

|\arg S(\pm, w_1) - \arg S(\pm, w_2)| \leq \frac{1}{2} |\arg w_1 - \arg w_2| + 0.3 \tag{21}

Recall now that S(\pm, z) map \mathbb{R}^d into \mathbb{R}^d. This means that we can iterate the above result starting from \( z_1, z_2 \in \mathbb{R}^d \). Using the notation

\[ \omega = (e_1, e_2, e_3, \ldots, e_n, \ldots), \tag{22} \]

and assuming \( n \geq 3 \), we readily discover

\[
|\arg \{\lambda + e_{n-2} \sqrt{(\lambda + e_{n-1} \sqrt{(\lambda + e_n \sqrt{z_1})})}\} \\
- \arg \{\lambda + e_{n-2} \sqrt{(\lambda + e_{n-1} \sqrt{(\lambda + e_n \sqrt{z_2})})}\}| \\
\leq \beta = 1.25 < \pi/2. \tag{23}
\]
Similarly, carrying out further iterations, we find

$$|\arg S_m(\omega, z_1) - \arg S_m(\omega, z_2)| \leq \beta$$

(24)

for all $m \geq 3$, for any $\omega \in \Omega$.

Again let us suppose $w_1, w_2 \in R_d$, but now with $|\arg w_1 - \arg w_2| \leq \beta$. Then

$$|S(\pm, w_1) - S(\pm, w_2)| = \frac{|w_1 - w_2|}{\sqrt{w_1} + \sqrt{w_2}} \leq \frac{1}{\sqrt{\cos \beta/2}} \frac{1}{\sqrt{w_1} + \sqrt{w_2}} \leq 0.87 = r < 1.$$  

It follows that for $n \geq 3$

$$|S_{n-3}(\omega, w_1) - S_{n-3}(\omega, w_2)| \leq 2dr^{n-3}$$

(26)

since $|w_1 - w_2| \leq 2d$. Finally we take $w_1$ and $w_2$ to be the two numbers whose arguments are compared in (23), and this yields (17) for $n \geq 3$. Since $r < 1$ and $|S_n(\omega, z_1) - S_n(\omega, z_2)| \leq 2d$ for all $n \in \{1, 2, \cdots\}$, the restriction $n \geq 3$ can be dropped.

Q.E.D.

**THEOREM 13** Let $0 < \lambda < 0.2$. Then $S(\omega)$ exists and lies in $R_a$ for all $\omega \in \Omega$.

**Proof.** For $0 < \lambda < 0.2$, $a \in R_a$. Let $m \geq n$ and note that
\[ |S_n(\omega, a) - S_m(\omega, a)| = |S_n(\omega, a) - S_n(\omega, S_{m-n}(\omega, a))| \] (27)

for some \( \omega \in \Omega \). Moreover \( S_{m-n}(\omega, a) \in R_a \) by the remark following Lemma 12. Hence Lemma 14 applies with \( z_1 = a \) and \( z_2 = S_{m-n}(\omega, a) \), yielding

\[ |S_n(\omega, a) - S_m(\omega, a)| \leq 2ar^{n-3}. \] (28)

Thus \( \{S_n(\omega, a)\}_{n=1}^{\infty} \) is a Cauchy sequence in the closed set \( R_a \). Q.E.D.

Our next objective is the identification of \( B_\lambda \) as defined in Theorem 11, and the set \( \{S(\omega) | \omega \in \Omega\} \).

We will continue to use the notation \( \sqrt{z} \) for the square root with positive axis branch cut. This notation is not compatible with our usage of the square root in the functional equation for \( F^{(2)}(z) \), and in the iterative equations for the \( f^{(n)}(z)\)'s. However, both \( F^{(2)}(z) \) and \( f^{(n)}(z) \) map the upper half plane into itself and so, consistently with our present convention we can write

\[
F^{(2)}(z) = \begin{cases} 
\lambda + \sqrt{F^{(2)}(z^2)} = S(\lambda, F^{(2)}(z^2)), & \text{when } 0 \leq \arg z < \pi, \\
\lambda - \sqrt{F^{(2)}(z^2)} = S(-\lambda, F^{(2)}(z^2)), & \text{when } \pi \leq \arg z < 2\pi,
\end{cases}
\] (29)

and for \( n \in \{1, 2, 3, \ldots\} \)

\[
f^{(n)}(z) = \begin{cases} 
\lambda + \sqrt{f^{(n-1)}(z^2)} = S(\lambda, f^{(n-1)}(z^2)), & \text{when } 0 \leq \arg z < \pi, \\
\lambda - \sqrt{f^{(n-1)}(z^2)} = S(-\lambda, f^{(n-1)}(z^2)), & \text{when } \pi \leq \arg z < 2\pi
\end{cases}
\] (30)
THEOREM 14. Let $0 < \lambda < 0.2$ and $0 < \theta < 2\pi$. Write

$$\theta = 2\pi \sum_{j=1}^{\infty} d_j / 2^j$$

where each $d_j$ belongs to \{0,1\}. Let

$$e_j = \begin{cases} +1 & \text{if } d_j = 0, \\ -1 & \text{if } d_j = 1, \end{cases}$$

and let $\omega = (e_1, e_2, e_3, \ldots)$. Then

$$\lim_{n \to \infty} F_n(e^{i\theta}) = S(\omega),$$

and

$$\lim_{r \to 1^+} F_\lambda(re^{i\theta}) = S(\omega).$$

Proof. Notice that for some values of $\theta$ there are two expansions (31), one involving infinitely many zeros and one involving infinitely many ones. In these cases there correspond two distinct elements $\omega_1$ and $\omega_2$ in $\Omega$, and for the theorem to make sense it must be true that $S(\omega_1) = S(\omega_2)$. But this is just what is implied by Lemma 13. Accordingly, without loss of generality, we can assume that $\omega$ contains infinitely many +1's.

Let $z = r e^{i\theta}$. Then it follows that $0 \leq \arg z^{2^{j-1}} < \pi$ precisely when $d_j = 0$. Thus, upon iterating (29) and (30) respectively, we obtain

$$F_\lambda(z) = S_n(\omega, F_\lambda(z^{2^n})) \text{ for } r > 1,$$

(35)
and

\[ f_n(z) = S_n(\omega, f_0(z^{2^n})) \text{ for } r \geq 1. \quad (36) \]

If \( r = 1 \) then \( f_0(z^{2^n}) \in R_a \) and we can proceed to the limit in (36) with the aid of Lemma 14 and Theorem 13, yielding (33).

Since the images of the exterior of the unit circle under \( \{f_n(z)\}_{n=0}^{\infty} \) increase to the image under \( F_A(z) \), it follows that \( \sup\{ |z| : z \in B_{\lambda} \} \leq a \), and

\[ \lim_{r \to 1} \sup_{|z| = r} |F_A(z)| \leq a. \quad (37) \]

To obtain (34), let \( \epsilon > 0 \) and choose \( d = 2a \) in Lemma 14. Pick \( n \) so that

\[ 4a(0.87)^{n-3} < \epsilon. \]

Furthermore (76) implies that there exists \( \rho > 1 \) such that

\[ |F_A(\omega)| \leq 2a \text{ when } 1 < |\omega| < \rho. \quad (38) \]

From the normalization of \( F_A(z) \) at infinity we know that for \( |z| > 1 \) and large enough \( n \), \( |F_A(z^{2^n})| > c \). Hence by (35) and Section 3.1 equation (12)

\[ |F_A(z)| \geq c \text{ when } |z| > 1. \quad (39) \]

Combining (38) and (39) we find that \( F_A(z^{2^n}) \in R_{2a} \) when \( 1 < |z| < \rho^{1/2^n} \).

We can now combine (35), Lemma 14 and Theorem 13 to provide

\[ |F_A(re^{i\theta}) - S(\omega)| < \epsilon \text{ when } 1 < r < \rho^{1/2^n}. \quad (40) \]

Q.E.D.
Next we extend the allowed values of $\lambda$ from the interval $0 < \lambda < 0.2$ to the set $L = \{ \lambda \in \mathbb{C} \mid |\lambda| < k \}$. We begin with a modification of Lemma 8: we observe that the inductive definition of $\{ f_n(z) \}_{n=0}^{\infty}$ and the domain monotonicity apply also for $\lambda \in L$.

**Lemma 15** Let $\lambda \in L$ and $n \in \{0,1,2,\cdots\}$. Then there exists a well defined one-to-one analytic function $f_n : D_0 \to \mathbb{C}$ such that

\[ f_n(z) = b^{-(n+1)} z + \lambda + o(1/z), \]  

(41)

and

\[ f_n(z) = \begin{cases} 
\lambda + \sqrt{b} \ z & \text{if } n = 0, \\
\lambda + \sqrt{f_{n-1}(z^2)} & \text{if } n > 0.
\end{cases} \]  

(42)

It has the properties

\[ f_n(D_0) \subset \{ z \in \mathbb{C} \mid |z| > c \}, \]  

(43)

and

\[ f_n(D_0) \supset f_{n-1}(D_0) \text{ for } n > 0. \]  

(44)

**Proof.** The proof follows the same lines as the proof of Lemma 8. There the statements $S_n$ (ii)-(iv) were used only to show that $0 \not\in f_n(D_0)$ which is essential to the inductive definition $f_n(z) = \lambda + \sqrt{f_{n-1}(z^2)}$. Here the set inclusion (43) implies that same fact.

In order to prove (43) note that
\[ f_0(z) = \lambda \pm \sqrt{b^2 z^2} \quad \text{and} \quad |b z^2| > |b| > |c| \quad (45) \]

for \( z \in D_0 \). Now apply Lemma 12 inductively.

The other parts of the proof are the same as in Lemma 3. Q.E.D.

In order to make the dependence on \( \lambda \) clearer in what follows we shall use the notation

\[ F(\lambda, z) = F_\lambda(z) \quad \text{and} \quad f_n(\lambda, z) = f_n(z). \quad (46) \]

**Theorem 15** There exists a continuous function \( F: \mathbb{R} \times \overline{D}_0 \to \hat{\mathbb{C}} \), analytic in \( \lambda \), one-to-one and analytic in \( z \) for \( z \in D_0 \), such that

\[ F(\lambda, z) = z + \lambda + O(1/z) \quad (47) \]

and

\[ F(\lambda, z) = \lambda + \sqrt{F(\lambda, z^2)}. \quad (48) \]

In particular, \( F_\lambda \) maps \( D_0 \) conformally onto \( D_\lambda \) and \( T_\lambda^{-1} B_\lambda = B_\lambda \) where \( B_\lambda \) denotes the boundary of \( D_\lambda \), for all \( \lambda \in \mathbb{R} \).

**Proof.** We can again parallel the proof of Theorem 10 to define

\[ F(\lambda, z) = \lim_{n \to \infty} f_n(\lambda, z) \quad \text{for} \quad \lambda \in \mathbb{R}, \ z \in D_0. \quad (49) \]

In this way we establish that \( F(\lambda, z) \) is one-to-one and analytic in \( z \) for \( z \in D_0 \), and that it obeys (47) and (48).
Now observe that for \( \lambda \in L \) the functions in the sequence \( \left\{ f_n(\lambda, z) \right\}_{n=0}^{\infty} \) as functions of \( \lambda \) are analytic, and their images omit the disk \( \{ z \in \mathbb{C} | |z| < c \} \). Hence they constitute a normal family. This means that they have an infinite subsequence which converges to some function \( H(\lambda, z) \) which, among its other properties, is analytic in \( \lambda \) for \( \lambda \in L \), for each fixed \( z \in \overline{D}_0 \). Suppose that there are in fact two different subsequences convergent to two different functions \( H_1 \) and \( H_2 \). Since the whole sequence is convergent to a single limit for \( 0 < \lambda < 0.2 \), \( H_1 \) and \( H_2 \) agree on a set which contains a limit point. Hence by Vitali's Theorem \( H_1 \equiv H_2 \). Thus we have the existence of

\[
H(\lambda, z) = \lim_{n \to \infty} f_n(\lambda, z)
\]

analytic in \( \lambda \) for \( \lambda \in L \) for each fixed \( z \in \overline{D}_0 \). Moreover \( H(\lambda, z) = F(\lambda, z) \) for \( z \in \overline{D}_0 \).

Next consider \( \lim_{r \to 1^+} F(\lambda, re^{i\theta}) \). For \( 0 < \lambda < 0.2 \) this limit exists and equals

\[
\lim_{r \to 1^+} f_n(\lambda, re^{i\theta}) = H(\lambda, e^{i\theta}).
\]

Thus, by again applying Vitali's Theorem, we have that the limit exists for all \( \lambda \in L \) and

\[
\lim_{r \to 1^+} F(\lambda, e^{i\theta}) = \lim_{n \to \infty} f_n(0, e^{i\theta}). \tag{51}
\]

It now follows that as a function of \( z \), \( F(\lambda, z) \) is a one-to-one analytic function on \( \overline{D}_0 \) with well-defined boundary values. By continuity the functional equation for \( F \) also applies on the boundary.

Q.E.D.
It turns out that the function \( \frac{\partial F}{\partial \lambda} (0, z) \) is easy to calculate. We use the fact that \( \frac{\partial F}{\partial \lambda} = \lim_{n \to \infty} \frac{\partial f_n}{\partial \lambda} \) and, for \( n > 0 \),

\[
\frac{\partial f_n}{\partial \lambda} (\lambda, z) = 1 + \frac{1}{\sqrt{f_n-1(z^2)}} \left( 1 + \frac{1}{\sqrt{f_{n-2}(z^4)}} \left( 1 + \cdots \left( 1 + \frac{1}{\sqrt{f_0(z^{2n})}} \right) \right) \right)
\]

(52)

Then since \( f_n(0, 0) = f_n(0, z) = z \) (note that \( b = 1 \) when \( \lambda = 0 \)) it follows that

\[
\frac{\partial f_n}{\partial \lambda} (0, z) = 1 + \frac{1}{2z} \left( 1 + \frac{1}{2z^2} \left( 1 + \cdots \left( 1 + \frac{1}{2z^{2n-1}} \right) \right) \right)
\]

(53)

and

\[
\frac{\partial F}{\partial \lambda} (0, z) = 1 + \sum_{j=1}^{\infty} \frac{1}{2^j z^{2j-1}}
\]

(54)

This series has the expected form. It converges uniformly for \( |z| \geq 1 \) and is analytic for \( |z| > 1 \). On the other hand it is a gap series: only the coefficients of \( (1/z)^{a_j} \) are nonzero with \( a_j = 2^{j-1} \) and \( \lim_{j \to \infty} (a_{j+1}/a_j) = 2 > 1 \).

Hence the unit circle is the natural boundary of the domain of the analytic function. In particular, we find that

\[
\lim_{r \to 1^+} \frac{\partial^2 F}{\partial \theta^{16}} (0, r e^{i \theta}) = \infty \quad \text{for} \quad \theta = 2\pi/2^n
\]

(55)

because for \( r > 1 \)

\[
\frac{\partial^2 F}{\partial \theta (0, re^{i \theta})} = \sum_{j=1}^{n-1} \left( \frac{1-2^j}{2^j} \right) \frac{1}{(re^{i \theta})^2^j} + \sum_{j=n}^{\infty} \frac{1-2^j}{2^j} \frac{1}{r^{2j}}
\]

(56)

The last sum blows up as \( r \to 1^+ \).
Now let

\[ G_\lambda(\theta) = F(\lambda, e^{i\theta}). \]  

Then Theorem 15 tells us among other facts that \( G_\lambda(\theta) \) is continuous in \( \theta \in \mathbb{R} \) for each \( \lambda \in L \). In fact we can say more.

**THEOREM 16.** Let \( M \) be any compact subset of \( L \). Then the family of functions \( \{ G_\lambda(\theta) | \lambda \in M \} \) is uniformly Hölder continuous in \( \theta \in \mathbb{R} \). That is, there exist positive constants \( K \) and \( \alpha \) such that

\[ |G_\lambda(\theta) - G_\lambda(\phi)| \leq K|\theta - \phi|^{\alpha} \]  

for all real \( \theta \) and \( \phi \), and all \( \lambda \in M \).

**Proof:** Since \( c > \frac{1}{2} \) for all \( \lambda \in L \) we can choose \( \beta \) with \( 0 < \beta < \pi/2 \) and \( R > 1 \), such that

\[ 2 \sqrt{c} \cos(\beta/2) > R > 1 \]  

for all \( \lambda \in M \).

Since \( G_\lambda(\theta) \) is continuous in \( \lambda \) and \( \theta \) we can choose \( \gamma > 0 \) such that

\[ |\arg(G_\lambda(\theta)/G_\lambda(\phi))| < \beta \text{ if } |\theta - \phi| < \gamma \]  

with \( \lambda \in M \).

Let \( \theta \) and \( \phi \) be real with \( |\theta - \phi| < \gamma \). Then there exists an integer \( n \geq 0 \) such that

\[ \gamma/2^{n+1} \leq |\theta - \phi| < \gamma/2^n. \]
Hence

\[ |G_\lambda(\theta) - G_\lambda(\phi)| = \left| G_\lambda(2\theta) - G_\lambda(2\phi) \right| \]

\[ = \left| \frac{G_\lambda(2\theta) - G_\lambda(2\phi)}{\sqrt{G_\lambda(2\theta)} + \sqrt{G_\lambda(2\phi)}} \right| \]

\[ \leq \frac{|G_\lambda(2\theta) - G_\lambda(2\phi)|}{R} \]  

(62)

Iterating we find

\[ |G_\lambda(\theta) - G_\lambda(\phi)| \leq \frac{|G_\lambda(2^n\theta) - G_\lambda(2^n\phi)|}{R^n} \leq \frac{K_2}{R^n} \]  

(63)

where \( K_2 \) is the maximum diameter of \( B_\lambda \) for \( \lambda \in \mathbb{M} \). Using (61) with \( \alpha = \frac{\ln R}{\ln 2} \) and \( K_2 = K_1(2/\gamma) \) we have

\[ |G_\lambda(\theta) - G_\lambda(\phi)| \leq K_2 |\theta - \phi|^\alpha \text{ for } |\theta - \phi| < \gamma \]  

(64)

To allow for \( |\theta - \phi| \geq \gamma \) we replace \( K_2 \) by

\[ K = \max\{K_2, K_1 \gamma^{-\alpha}\}. \]  

(65)

Q.E.D.
§4. Pictures Related to $B_\lambda$

Here we present pictorially the results of some calculations which concern the structure of $B_\lambda$. The theory of previous sections provides the basis for the computations and for the interpretation of the resulting figures.

Figures 3, 4, 5, and 6, represent the set $B_\lambda$ for $0 \leq \lambda \leq 2$. These drawings are not exact, but are based upon similar pictures in which about five hundred points belonging to the set $B_\lambda$ were plotted. The plotted points were all of the form $S_{50}(\omega, a)$, see Sec. 3.2 equation (7), and $\omega \in \Omega$ was chosen at random. It was found that the pictures obtained were not significantly different when a negative axis branch cut was used in place of a positive axis branch cut, in the evaluation of the square root. Notice that, although in the figures $B_\lambda$ is represented by a collection of points, Theorem 14 provides that for $\lambda = 0.001$, and $\lambda = 0.1$, it is a continuous curve.

An alternative view of $B_\lambda$ is indicated in Figures 7 and 8, where we have plotted the boundaries of the sets $f_n(D_0)$ for $n \in \{1, 2, 3, 4\}$, at two different values of $\lambda$. Recall that $f_1(D_0) \subset f_2(D_0) \subset f_3(D_0) \subset f_4(D_0) \subset \cdots$, and that the boundary of the set obtained in the limit is $B_\lambda$, see Lemma 8 and Theorem 10.

In Figures 9, 10, and 11, we track the two-cycle, the three-cycles, and the four-cycles of $T_\lambda$. These cycles belong to the set $B_\lambda$, and the various values of $\lambda$ at which they first become real are the critical values at which the corresponding real cycles first occur. The two-cycle which is followed in Figure 9 corresponds to the two points represented by
and
\[
\lambda + \sqrt{\lambda - \sqrt{\lambda + \sqrt{\lambda + \sqrt{\lambda + \cdots}}}}
\]
and
\[
\lambda - \sqrt{\lambda + \sqrt{\lambda - \sqrt{\lambda - \sqrt{\lambda - \cdots}}}}
\]
for \(0 \leq \lambda < 0.75\). For this range of \(\lambda\) the two points have nonzero imaginary parts, and the positive axis branch cut is used in the evaluation of the square roots.

The three-cycle in Figure 10 which commences at the points labelled 1, 2, and 3, corresponds respectively to the three members of \(B_\lambda\) indicated by
\[
\lambda + \sqrt{\lambda + \sqrt{\lambda + \sqrt{\lambda + \sqrt{\lambda + \cdots}}}}
\]
and
\[
\lambda - \sqrt{\lambda + \sqrt{\lambda - \sqrt{\lambda - \sqrt{\lambda - \cdots}}}}
\]
and
\[
\lambda - \sqrt{\lambda + \sqrt{\lambda + \sqrt{\lambda + \sqrt{\lambda + \cdots}}}}
\]
The complex conjugate three cycle labelled 4, 5, and 5, is obtained by reversing the signs in these three expressions. Again, the positive axis branch cut is used.

The self-conjugate four-cycle in Figure 11 commences when \(\lambda = 0\) at the points indicated 1, 2, 3 and 4, and corresponds respectively to
\[ \lambda + \sqrt{\lambda} + \sqrt{\lambda - \sqrt{\lambda} - \cdots}, \]

\[ \lambda + \sqrt{\lambda} - \sqrt{\lambda - \sqrt{\lambda} + \cdots}, \]

\[ \lambda - \sqrt{\lambda - \sqrt{\lambda + \sqrt{\lambda} + \cdots}}, \]

and

\[ \lambda \sqrt{\lambda + \sqrt{\lambda} + \sqrt{\lambda} - \cdots}. \]

The four-cycle which commences when \( \lambda = 0 \) at the points indicated 5, 6, 7, and 8, corresponds to

\[ \lambda + \sqrt{\lambda + \sqrt{\lambda} + \sqrt{\lambda} - \cdots}, \]

\[ \lambda + \sqrt{\lambda} + \sqrt{\lambda - \sqrt{\lambda} + \cdots}, \]

\[ \lambda + \sqrt{\lambda - \sqrt{\lambda + \sqrt{\lambda} + \cdots}}, \]

and

\[ \lambda - \sqrt{\lambda + \sqrt{\lambda} + \sqrt{\lambda} + \cdots}. \]
The various points defined above in terms of the continued square root with the positive axis branch cut can be re-expressed using the negative axis branch cut. Let \((e_1, e_2, e_3, \ldots)\) be the sequence of plus and minus ones associated with a point in \(B_\lambda\) with \(-\lambda < \lambda < 2\), with positive axis cut, and let \((s_1, s_2, s_3, \ldots)\) be the sequence for the same point with negative axis cut. Then \(s_1 = e_1^2, s_2 = e_2^2, \ldots s_j = e_j^2, \ldots\). Conversely, given \((s_1, s_2, s_3, \ldots)\), we can invert these equations by choosing \(e_1\) arbitrarily and then \(e_2 = e_1 s_1, e_3 = e_2 s_2, \ldots, e_j = e_{j-1} s_{j-1}, \ldots\). This transformation should be compared with the one described at the end of Section 2.2.

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FIGURE 1: The boundaries $\Gamma_n$ for $n = 0, 1, 2$, and 3, are drawn unbroken; the dots lie upon the coordinate axes. In general, $\Gamma_{n+1}$ contains $2^n$ new arcs not in $\Gamma_n$, and $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \ldots$.\[\]

FIGURE 2: The boundaries of $f_3(D_0)$ and $f_3^*(D_0)$ are superimposed, and sixteen components are thereby defined. The closure of the component labeled $j$ contains the part of $B_\lambda$ which corresponds to $\{e^{i(j-1)\pi/16} \leq 0 \leq j\pi/16\}$.\[\]

FIGURE 3: Representation of $B_\lambda$ for $\lambda = 0.001$, 0.1, and 0.3. The crosses indicate points in the complex plane which belong to $B_\lambda$. These drawings are based upon accurate plots of five hundred points in $B_\lambda$, picked at random.\[\]

FIGURE 4: Representation of $B_\lambda$ for $\lambda = 0.5$, 0.7, and 0.9. See the caption of Fig. 3.\[\]

FIGURE 5: Representation of $B_\lambda$ for $\lambda = 1.0$, 1.2, and 1.4. See the caption of Fig. 3.\[\]

FIGURE 6: Representation of $B_\lambda$ for $\lambda = 1.7$, 1.8, and 2.10. See the caption of Fig. 3.\[\]

FIGURE 7: Successive inclusion domains for $B_\lambda$ when $\lambda = 0.75$. The boundary indicated by $n \in \{1, 2, 3, 4\}$ is the image of the unit circle under $f_n(z)$, as defined in Lemma 8. The sequence of boundaries converges to $B_\lambda$ in the manner described in Section 3.1.
FIGURE 8: Same as Figure 7, but here $\lambda = 1.25$.

FIGURE 9: Shown is the path followed in the complex plane by the two-cycle of $T_\lambda$. At $\lambda = 0$ the two members of the cycle are the points labeled 1 and 2. They arrive on the real axis, at the point marked $\Lambda$, when $\lambda = 3/4$. It is at this juncture that the real two-cycle first appears. Their locations when $\lambda = 1$ are indicated.

FIGURE 10: The map $T_\lambda$ possesses two three-cycles. Shown are the paths followed by these three-cycles as $\lambda$ varies from 0 to 1.75. At $\lambda = 0$, one of the three cycles consists of the points labelled 1, 2, 3, and the other is its complex conjugate with corresponding points labelled 4, 5, 6. We have $T_01 = 2$, $T_02 = 3$, $T_03 = 1$, $T_04 = 5$, $T_05 = 6$, and $T_06 = 4$. As $\lambda$ increases, the points move along the indicated paths in the directions of the arrows, and when $\lambda$ reaches 1 they are located at the positions indicated by $\Xi$. When $\lambda = 1.75$ the two three-cycles arrive on the real axis at the positions indicated by $\Lambda$, in complex conjugate pairs. It is at this juncture that the real three-cycles first appear.

FIGURE 11: Shows four-cycles of $T_\lambda$. At $\lambda = 0$ the self-conjugate four-cycle is located at the points labeled 1, 2, 3, and 4. These points have the values $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{8\pi i/5}$, and $e^{6\pi i/5}$ respectively. They arrive upon the real axis in complex conjugate pairs at two of the points marked $\Lambda$, when $\lambda = 1.25$. Another four-cycle commences at $\lambda = 0$ at the points labeled 5, 6, 7, 8. The points have
the values $e^{\frac{2\pi i}{15}}$, $e^{\frac{4\pi i}{15}}$, $e^{\frac{8\pi i}{15}}$, and $e^{\frac{16\pi i}{15}}$ respectively. When $\lambda = 1.25$ they have shifted along the indicated paths to locations marked A. There exists also the complex conjugate of the latter four cycle, but is not shown in the figure. These two complex conjugate four cycles join the real axis when approximately $\lambda = 1.95$. This accounts for all of the four-cycles of the map $T_\lambda: \mathbb{C} \to \mathbb{C}$. 
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Figure 6